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APPLICATION OF OPERATIONAL METHODS TO SWITCHING PROBLEMS.

BY B. M. BROWN.

IN the interesting new book by Carslaw and Jaeger, *Operational Methods in Applied Mathematics*, a point is raised which does not seem to have been explained adequately in this or any other book on this subject that I have seen. This concerns the application of initial conditions to switching problems in electric circuit theory. Reference is made to this deficiency in a review of this book in the *Gazette*, XXVI, No. 268 (February 1942), pp. 63-64, and the present article is a suggested way of overcoming it.

The solution of ordinary linear differential equations by operational methods can be introduced by many different methods. When it comes to practical applications, however, the various methods are very similar. The operator d/dt has to be replaced by p, provided that the function on which it operates vanishes when $t=0$. The equation is then solved algebraically for the operational form of the dependent variable, and the latter is interpreted by means of a number of standard forms. An immediate extension of this method is applicable to simultaneous equations.

The cases in which operands do not vanish when $t=0$ are treated by means of certain simple rules. A convenient process is to make the operands vanish initially by subtraction of the appropriate constants.† Thus if $y=y_0$ and $dy/dt=y_1$, etc., when $t=0$, the O.F. of dy/dt is $p(y^*-y_0)$, and of d^2y/dt^2, $p\{p(y^*-y_0)-y_1\}$, etc.‡

In many books on the subject, the initial vanishing of the operand is stressed in the early chapters, and the initial conditions are stated carefully. Later on they are given only by implication. Especially is this so in problems on electric circuits. The initial conditions in those cases are usually those arising from the closing of a switch at zero time in a circuit initially quiescent. Without further consideration, the inductive terms of the type LdI/dt are replaced by LpI^*, thus implying that I vanishes when $t=0$.

† It is assumed that the Heaviside operational forms are used, that is to say that the O.F. of exp kt is $p/(p-k)$, etc. A slight modification is necessary if Laplace forms are used.

‡ A star will be used to denote operational forms.

In the simpler problems this assumption is correct, although unjustified. But in certain types of coupled circuit the expressions obtained for the currents do *not* vanish initially, and it must be deduced either that the solution is that of another problem, or that the result is correct and the process justifiable. Actually the latter is the case and the justification of this will appear presently. What happens is that during the small time interval in which the switch is being closed, the currents change from zero to the initial values as given by the operational method.

Let us consider the general case of a network which is initially quiescent. Let us further assume that at zero time a number of switches are closed, thus applying to the circuit certain external electromotive forces and allowing initially charged condensers to become active. The solution is effected by means of the appropriate number of linear simultaneous differential equations of the types

$$\Sigma L\frac{dI}{dt}+\Sigma RI+\Sigma\frac{Q}{C}=E, \qquad (1)$$

$$\frac{dQ}{dt}=I. \qquad (2)$$

Let the time taken to close the switches be τ. More precisely, it is convenient to regard the interval as being from $t=-\tau$ to $t=0$. It is assumed that τ is infinitesimal. Let the values of I, Q and E at the end of this interval (that is, at time $t=0$) be denoted by i, q and e respectively. The operational forms of (1) and (2) can then be written

$$p(\Sigma LI^*-\Sigma Li)+\Sigma RI^*+\Sigma Q^*/C=E^*, \qquad (3)$$

$$p(Q^*-q)=I^*. \qquad (4)$$

Before we can proceed we must overcome the difficulty that i and q are unknown. It will be assumed that no point of the network ever has infinite potential. This is equivalent to saying that each of the four terms of (1) must remain finite in all circumstances.

During the interval τ the currents may have finite changes in their values, which will imply that dI/dt is infinite during this interval. (Although dI/dt may be infinite, $\Sigma L\,dI/dt$ must be finite, since the potential cannot be infinite). Thus it cannot be assumed that i is zero. If the resistance vanishes, I may even become infinite momentarily. This is the case of the impulsive current, which will be considered later.

Before and during the interval τ, (1) must be replaced by

$$\Sigma L\frac{dI}{dt}+\Sigma RI+\Sigma\frac{Q}{C}=E-E', \qquad (5)$$

where E' is the potential difference between the electrodes of the switch. During the interval τ, E' will fall from $E_0-\Sigma Q_0/C$ to zero, where E_0 and Q_0 are the values of E and Q when $t=-\tau$.

Integrate (5) from $t=-\tau$ to $t=0$.

$$\Sigma Li+\int_{-\tau}^{0}\{\Sigma RI+\Sigma Q/C-E+E'\}\,dt=0.$$

Since the integrand is finite, the integral must be infinitesimal. Hence

$$\Sigma Li=0. \qquad (6)$$

Thus the first term of (3) may be written $p\Sigma LI^*$, and the process described above is justified.

Considering now equations (2) and (4), we see that unless there is an impulsive current during τ, the change in charge on the condensers will be infinitesimal, so that (4) may be written

$$p(Q^* - Q_0) = I^*. \qquad (7)$$

Let us see what will be the effect of using (7) when there is an impulsive charging current (there may be no way of deciding in advance whether this is or is not the case). Instead of (4), the equation

$$p(Q^* - q) + p(q - Q_0) = I^*$$

is used. Thus the expression for I^* will contain an extra term $p(q - Q_0)$. Now $q - Q_0$ is the impulse of the initial charging current, so that if (7) is used this quantity is indicated in the solution as well as the value of the current at any subsequent time.

Summarising, we can say that *if the operational forms of* (1) *and* (2) *for an initially quiescent network are expressed in the forms*

$$p\Sigma LI^* + \Sigma RI + \Sigma Q^*/C = E^*, \qquad (8)$$

$$p(Q^* - Q_0) = I^*, \qquad (9)$$

the correct expressions for charge and current are obtained. A term of the type Ap in the expression for I^* must be interpreted as an initial charging current of impulse A.

It is desirable to write down the operational equations (8) and (9) before eliminating I or Q. If this is done, the same method can be used for all types of problem.

Example 1. *The circuit of Fig.* 1 *is initially quiescent and the condensers are uncharged. If the self and mutual inductances are all equal to L, the condensers have capacity C, and a constant E.M.F. E_0 is applied, find the currents and charges.*

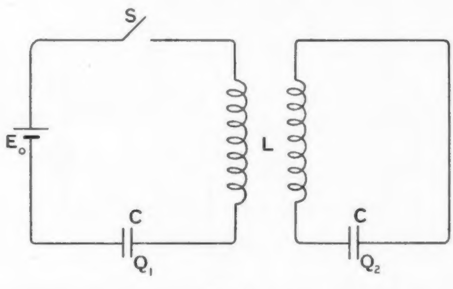


FIG. 1.

The operational equations are

$$p(LI_1^* + LI_2^*) + Q_1^*/C = E_0, \quad pQ_1^* = I_1^*;$$

$$p(LI_1^* + LI_2^*) + Q_2^*/C = 0, \quad pQ_2^* = I_2^*.$$

Solving in the usual way gives

$$I_1^* = \frac{E_0}{4L}\frac{p}{p^2 + 1/2LC} + \frac{E_0 C}{2} p,$$

$$I_2^* = \frac{E_0}{4L}\,\frac{p}{p^2+1/2LC} - \frac{E_0 C}{2}\,p,$$

$$Q_1^* = \frac{E_0 C}{2}\,\frac{p^2+1/LC}{p^2+1/2LC},$$

$$Q_2^* = -\frac{E_0 C}{2}\,\frac{p^2}{p^2+1/2LC}.$$

These are interpreted as follows :

$$I_1 = \frac{E_0}{2}\sqrt{\left(\frac{C}{2L}\right)}\sin\frac{t}{\surd(2LC)} \quad \text{with initial impulse } \tfrac{1}{2}E_0 C,$$

$$I_2 = \frac{E_0}{2}\sqrt{\left(\frac{C}{2L}\right)}\sin\frac{t}{\surd(2LC)} \quad \text{with initial impulse } -\tfrac{1}{2}E_0 C,$$

$$Q_1 = E_0 C\left(1 - \tfrac{1}{2}\cos\frac{t}{\surd(2LC)}\right),$$

$$Q_2 = -\frac{E_0 C}{2}\cos\frac{t}{\surd(2LC)}.$$

The process explained above can be generalised so as to be applicable to switching problems in which the circuit is not necessarily quiescent initially. In this case, if $I = I_0$ when $t = -\tau$, (6) must be replaced by

$$\Sigma Li = \Sigma LI_0.$$

Example 2. *In the circuit shown in Fig.* 2 *the switch S is opened at zero time, when the current flowing is I_0 and the charge on the condenser is Q_0. Find the current at any subsequent time.*

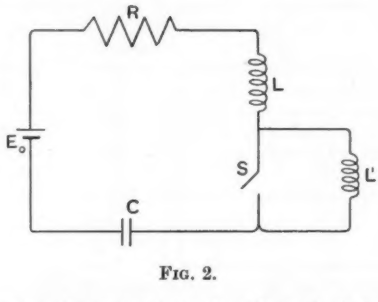


FIG. 2.

The effect of opening the switch is to introduce the inductance L' in series with the other impedances. The operational equations will be

$$p\{(L+L')I^* - LI_0\} + RI^* + Q^*/C = E_0,$$
$$p(Q^* - Q_0) = I^*.$$

Eliminating Q^*,

$$\{p(L+L') + R + 1/pC\}I^* = E_0 + pLI_0 + Q_0/C.$$

I can then be obtained by the usual method. B. M. B.

AN OPERATIONAL METHOD OF SOLVING LINEAR DIFFERENTIAL EQUATIONS.

By H. V. Lowry.

In this article the method of solution is applied to Bessel's equation, but it can be used in the same way to solve other linear equations.

If we put

$$x = e^t, \text{ so that } x\frac{d}{dx} = \frac{d}{dt} = \Delta,$$

in Bessel's equation of zero order

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y = 0,$$

it becomes

$$(\Delta^2 + e^{2t})\,y = 0.$$

Now let

$$\Delta^2 u_1 = 0,$$
$$\Delta^2 u_2 = -e^{2t}u_1,$$
$$\Delta^2 u_3 = -e^{2t}u_2,$$

and so on. Then

$$(\Delta^2 + e^{2t})(u_1 + u_2 + \ldots + u_n) = e^{2t}u_n.$$

Hence $y = \sum\limits_1^\infty u_n$ is a solution of Bessel's equation so long as $e^{2t}u_n$ tends to zero as n tends to infinity. We shall assume that this condition is satisfied.

Now

$$u_2 = -\frac{1}{\Delta^2}e^{2t}u_1 = -e^{2t}\frac{1}{(\Delta+2)^2}u_1,$$

$$u_3 = -\frac{1}{\Delta^2}e^{2t}u_2 = -\frac{1}{\Delta^2}e^{4t}\frac{1}{(\Delta+2)^2}u_1 = e^{4t}\frac{1}{(\Delta+2)^2(\Delta+4)^2}u_1,$$

and so on. In general

$$u_n = (-1)^{n-1}e^{(2n-2)t}\frac{1}{(\Delta+2)^2(\Delta+4)^2(\Delta+2n-2)^2}u_1.$$

But

$$u_1 = A + Bt.$$

Hence

$$u_n = (-1)^{n-1}e^{(2n-2)t}\frac{1}{2^2.4^2\ldots(2n-2)^2}A$$

$$+(-1)^{n-1}e^{(2n-2)t}\frac{1}{2^2.4^2\ldots(2n-2)^2}\left\{1-\left(1+\frac{1}{2}+\ldots+\frac{1}{n-1}\right)\Delta\right\}Bt$$

$$=(-1)^{n-1}e^{(2n-2)t}\frac{1}{2^2.4^2\ldots(2n-2)^2}\left\{A+Bt-B\left(1+\frac{1}{2}+\ldots+\frac{1}{n-1}\right)\right\}$$

$$=(-1)^{n-1}x^{2n-2}\frac{1}{2^2.4^2\ldots(2n-2)^2}\left\{A+B\log x-B\left(1+\frac{1}{2}+\ldots+\frac{1}{n-1}\right)\right\}.$$

Hence the general solution for y is

$$y=\sum_1^\infty u_n=\sum_1^\infty\frac{(-1)^{n-1}x^{2n-2}}{2^2.4^2\ldots(2n-2)^2}\left\{A+B\log x-B\left(1+\frac{1}{2}+\ldots+\frac{1}{n-1}\right)\right\},$$

where there is no term in the last bracket for $n=1$. This is

$$y = AJ_0(x) + Y_0(x)$$

where $Y_0(x)$ is Neumann's Bessel function of the 2nd kind of zero order.

We will now apply the same method to Bessel's equation of order n, namely:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0.$$

Using the operator Δ, this becomes

$$(\Delta^2 - n^2 + e^{2t})\, y = 0.$$

Let

$$(\Delta^2 - n^2)\, u_1 = 0,$$
$$(\Delta^2 - n^2)\, u_2 = -e^{2t}u_1,$$
$$(\Delta^2 - n^2)\, u_3 = -e^{2t}u_2,$$

and so on.

Then

$$u_2 = -\frac{1}{\Delta^2 - n^2} e^{2t}\, u_1$$
$$= -e^{2t} \frac{1}{(\Delta + 2 + n)(\Delta + 2 - n)} u_1,$$
$$u_3 = -\frac{1}{\Delta^2 - n^2} e^{2t}\, u_2$$
$$= \frac{1}{\Delta^2 - n^2} e^{4t} \frac{1}{(\Delta + 2 + n)(\Delta + 2 - n)} u_1$$
$$= e^{4t} \frac{1}{(\Delta + 2 + n)(\Delta + 4 + n)(\Delta + 2 - n)(\Delta + 4 - n)} u_1.$$

In general

$$u_r = e^{(2r-2)t} \frac{(-1)^{r-1}}{(\Delta + 2 + n) \ldots (\Delta + 2r - 2 + n)(\Delta + 2 - n) \ldots (\Delta + 2r - 2 - n)} u_1.$$

But

$$u_1 = Ae^{nt} + Be^{-nt}.$$

Hence, remembering that $\Delta(e^{nt}) = ne^{nt}$,

$$u_r = Ae^{(2r-2+n)t} \frac{(-1)^{r-1}}{(2n+2) \ldots (2n + 2r - 2)\, 2 \ldots (2r-2)}$$
$$+ Be^{(2r-2-n)t} \frac{(-1)^{r-1}}{2 \ldots (2r-2)(-2n+2) \ldots (-2n + 2r - 2)}$$
$$= \frac{A(-1)^{r-1}x^{2r-2+n}}{2 \ldots (2r-2)(2n+2) \ldots (2n+2r-2)} + \frac{B(-1)^{r-1}x^{2r-2-n}}{2 \ldots (2r-2)(-2n+2) \ldots (-2n+2r-2)}.$$

The general solution is $y = \sum_1^{\infty} u_r$, which is $AJ_n(x) + BJ_{-n}(x)$.

If, however, n is an integer, $u_{n+1}, u_{n+2} \ldots$ have a zero factor in the denominator of the second term, so that after u_n the operator must be used in a different way.

Since we are only concerned with the change in the second term, we will call the parts of u_r derived from Ae^{nt} and Be^{-nt} Av_r and Bw_r respectively.

Then v_r is formed as before for all values of n, but w_r is formed differently after w_n.

Now

$$
\begin{aligned}
w_{n+1} &= -\frac{1}{\Delta^2 - n^2} e^{2t} w_n \\
&= -e^{2t} \frac{1}{(\Delta + 2 + n)(\Delta + 2 - n)} e^{(2n-2-n)t} \frac{(-1)^{n-1}}{2 \ldots (2n-2)(-2n+2) \ldots (-2)} \\
&= -e^{2t} \frac{(-1)^{n-1}}{2 \ldots (2n-2)(2n)(-2n+2) \ldots 2} \frac{1}{\Delta + 2 - n} e^{(n-2)t} \\
&= \frac{(-1)^n}{2 \ldots 2n \,.\, (-2n+2) \ldots 2} te^{nt} \\
&= kte^{nt} \text{ say.}
\end{aligned}
$$

Hence

$$
\begin{aligned}
w_{n+2} &= -\frac{1}{\Delta^2 - n^2} e^{2t} w_{n+1} = -\frac{1}{\Delta^2 - n^2} ke^{(n+2)t} t \\
&= -ke^{(n+2)t} \frac{1}{(\Delta + 2n + 2)(\Delta + 2)} t \\
&= -ke^{(n+2)t} \frac{1}{(2n+2)2} \left\{ 1 - \left(\frac{1}{2} + \frac{1}{2n+2} \right) \Delta + \text{higher powers of } \Delta \right\} t \\
&= -ke^{(n+2)t} \frac{1}{(2n+2)2} \left\{ t - \left(\frac{1}{2} + \frac{1}{2n+2} \right) \right\}.
\end{aligned}
$$

In general,

$$
\begin{aligned}
w_{n+p} &= \left\{ -\frac{1}{\Delta^2 - n^2} e^{2t} \right\}^{p-1} kte^{nt} \\
&= ke^{(2p-2)t} \frac{(-1)^{p-1}}{(\Delta + 2 + n) \ldots (\Delta + 2p - 2 + n)(\Delta + 2 - n) \ldots (\Delta + 2p - 2 - n)} e^{nt} t \\
&= ke^{(2p-2+n)t} \frac{(-1)^{p-1}}{(\Delta + 2 + n) \ldots (\Delta + 2p - 2 + n)(\Delta + 2 - n) \ldots (\Delta + 2p - 2 - n)} t \\
&= ke^{(2p-2-n)t} \frac{(-1)^{p-1}}{(2n+2) \ldots (2n+2p-2) 2 \ldots (2p-2)} \\
&\qquad \left\{ t - \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2p-2} - \frac{1}{2n+2} - \ldots - \frac{1}{2n+2p-2} \right) \right\} \\
&= kx^{2p-2+n} \frac{(-1)^{p-1}}{(2n+2) \ldots (2n+2p-2) 2 \ldots (2p-2)} \\
&\qquad \left\{ \log x - \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2p-2} - \frac{1}{2n+2} - \ldots - \frac{1}{2n+2p-2} \right) \right\}.
\end{aligned}
$$

Thus

$$
w_{n+p} = kv_p \left\{ \log x - \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2p+2} - \frac{1}{2n+2} - \ldots - \frac{1}{2n+2p-2} \right) \right\}.
$$

Hence

$$\sum_1^\infty w_r = \sum_{r=1}^{n} \frac{(-1)^{r-1}x^{2r-2+n}}{2 \ldots (2r-2)(-2n+2) \ldots (-2n+2r-2)}$$

$$+ k \sum_{p=1}^{\infty} v_p \left\{ \log x - \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2p-2} - \frac{1}{2n+2} - \ldots - \frac{1}{2n+2p-2} \right) \right\}$$

$$= \sum_{r=1}^{n} \frac{(-1)^{r-1}x^{2r-2+n}}{2 \ldots (2r-2)(-2n+2) \ldots (-2n+2r-2)} + kJ_n \log x$$

$$+ k \sum_{p=2}^{\infty} v_p \left\{ \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2p-2} - \frac{1}{2n+2} - \ldots - \frac{1}{2n+2p-2} \right) \right\}.$$

Therefore the complete solution for y is

$$y = J_n(A + kB \log x)$$

$$+ B \left[\sum_{r=1}^{n} \frac{(-1)^{r-1}x^{2r-2+n}}{2 \ldots (2r-2)(-2n+2) \ldots (-2n+2r-2)} \right.$$

$$\left. + k \sum_{p=2}^{\infty} v_p \left\{ \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2p-2} - \frac{1}{2n+2} - \ldots - \frac{1}{2n+2p-2} \right\} \right],$$

where $$k = \frac{(-1)^n}{2 \ldots 2n(-2n+2) \ldots 2},$$

and $$v_p = \frac{(-1)^{p-1}x^{2p-2+n}}{(2n+2) \ldots (2n+2r-2)(2) \ldots (2r-2)}.$$

The advantage of this method of solution is that the terms involving $\log x$ arise naturally in the general course of the solution. The method is really equivalent to the division method given in Note 1486 in No. 262 of the *Mathematical Gazette*, but it is much less cumbersome to apply. H. V. Lowry.

Since I wrote the above note I have found that the same method is used to find the solution of Bessel's equation of zero order in Prescott's *Applied Elasticity*, but it is not extended there to cover the equation of the nth order.

GLEANINGS FAR AND NEAR.

1406. Some of New York's leading doctors and physiologists, I hear, have had to deal with an astonishing increase in the number of physical and nervous ailments. These they attribute to worry over the war.

Many of their patients are suffering from headaches, digestive trouble or insomnia. They complain of:

Highly disturbed mental state ;	Inability to concentrate ;
Melancholia ;	Listlessness ;
Depression ;	Fatigue ;
Apprehension ;	Forebodings ;
Fears for the future ;	Hollowness ;
Uncertainty ;	Breathlessness ;
Sense of insecurity ;	Jittery nerves.

Jittering, to judge from London's experience, seems to vary *inversely* with the square of the distance.—" Peterborough " in the *Daily Telegraph*, June 26, 1940. [Per Dr. D. Pedoe.]

FURTHER NOTE ON THE MOTION OF A BODY WHOSE MASS IS CHANGING.

BY A. S. RAMSEY.

1. The note on this subject published in the *Gazette* of July 1941 was written in the hope that it might clear away a difficulty which I have good reason to know many students and teachers of dynamics find, when confronted by a problem of changing mass, where it is not obvious which, if indeed either, of the equations $M\,dV/dt = F$ or $d(MV)/dt = F$ applies. Among letters previously received on the subject I have found it asserted that the latter equation is "fundamental" and that results not in conformity with it must be wrong. I showed by three simple examples of changing mass that it is possible for one or other of these equations to be true or for neither to be true, and hoped that readers would feel satisfied to form the equation of motion for any such case directly, by the simple process of equating the change of momentum in time δt to the impulse of the force producing the change. Could there be a simpler procedure? Unfortunately readers in general are not so easily satisfied, and I seem to have stimulated an ambition to produce an equation of motion "of quite general validity" applicable to all such cases. My first correspondent was satisfied that his equation was of general validity, but when he applied it to my third example he came to the conclusion that the example must be wrong. And now in the *Gazette* for February 1942, p. 59, we have a note by Mr. R. Newing stressing the importance of a general equation, which he finds "implicit in Ramsey's work (and actually given in Palmer and Snell's *Mechanics*, p. 290, equation (2))". I quote this equation, as I propose to comment on it :

$$d(M\mathbf{V})/dt - \mathbf{u}\,dM/dt = \mathbf{F}. \qquad (1)$$

Here $\mathbf{V}$ denotes the velocity of the body of mass M at time t and $\mathbf{u}$ denotes the velocity of matter at the instant of its absorption (or ejection). Subject to these hypotheses this equation is obtained by the simple process of equating the change of momentum in time δt to the impulse of the force producing it. This principle, I need hardly say, is due to Newton. Mr. Newing claims that "this equation is actually of quite general validity", and if this merely means that it is true in all cases to which it can be applied no-one would be likely to dispute the statement ; but if it is a claim that from it can be deduced the correct equations of motion for all simple problems of changing mass, then it is an assertion that cannot be maintained.

2. Mr. Newing attempts to apply (1) to the solution of the three examples I gave in my former note. The first two are easy going, but the third presents a difficulty : *An engine of total mass M at time t is moving with velocity V and in each unit of time converts a mass m of fuel into a mass m' of smoke and gas and ejects it vertically.**

It is easily shown directly (*Gazette*, July 1941, p. 142) that the equation of motion is

$$M\,dV/dt - mV + m'V = F, \qquad (2)$$

* It has been represented to me that the words "ejects it vertically" are capable of the meaning that *after* ejection the smoke and gas are moving vertically. I agree ; but inasmuch as this would require a mechanical device whereby an engine, itself moving with variable acceleration, would be able to impart to its smoke a velocity equal and opposite to that of the atmosphere, I need hardly say that this was not the meaning I intended. My meaning, as shown in my solution, is that the engine ejects the smoke and gas through a vertical chimney and therefore they possess the horizontal velocity of the engine.

or, writing $m = -dM/dt$,

$$d(MV)/dt + m'V = F. \quad\text{.................................}(3)$$

In obtaining (3) from (2) we use the fact that $dM/dt = -m$, while Mr. Newing obtains (3) from (1) by using the relation $dM/dt = -m'$. He obtains this relation by assuming that m is greater than m' and that the mass $m - m'$ is carried along by the engine. The fact is that m', the mass of the products of combustion, is greater than the mass of fuel m. Thus $m - m'$ is negative and Mr. Newing's argument breaks down.

Further, as Mr. Newing points out, in *every* case in which $\mathbf{u} = \mathbf{V}$ equation (1) reduces to the simple form $M\,d\mathbf{V}/dt = \mathbf{F}$. So, since my example (iii) is a case in point, in which u, the velocity of the matter ejected, is the same as V the velocity of the body, if (1) were applicable the equation of motion would not be (3) but simply $M\,dV/dt = F$. But (3) cannot take this form unless $m' = m$, which alters the problem. We conclude that Example (iii) is a case to which equation (1) does not apply; and Mr. Newing's claim as to the "quite general validity" of equation (1) therefore needs some qualification.

3. I ought perhaps to have stated explicitly in the enunciation of Example (iii) that it is assumed that the changes take place in an atmosphere at rest, though this might be taken for granted as the contrary was not stated. But in order to exhibit further the inapplicability of Mr. Newing's equation, I will now consider a similar problem with rather more details, supposing that *the engine consumes fuel at the rate of m units of mass per second which combines with oxygen in an atmosphere whose component velocity is v in the direction of motion of the engine, thus producing a total mass m' per second, whereof a portion m'' is ashes which are ejected horizontally along the track with velocity $-v'$ relative to the engine, while the remainder $m' - m''$ consists of smoke and gas and is allowed to escape with the velocity of the engine.*

We now have

$$\text{momentum at time } t = MV + (m' - m)v\,\delta t,$$

where the second term is, to the first order, the momentum of the oxygen that enters into combination in time δt. In time δt the masses are redivided into $M - m\,\delta t$, $m''\delta t$ and $(m' - m'')\delta t$, and the momentum at time $t + \delta t$ is, to the first order,

$$(M - m\,\delta t)(V + \delta V) + m''(V - v')\delta t + (m' - m'')V\,\delta t,$$

where the second and third terms are the momenta of the ashes and the smoke and gas respectively.

By equating the gain in momentum to $F\,\delta t$ we get

$$M\,dV/dt - mV + m''(V - v') + (m' - m'')V - (m' - m)v = F;$$

and again $dM/dt = -m$, so that the equation reduces to

$$d(MV)/dt + mv + m'(V - v) - m''v' = F. \quad\text{.....................}(4)$$

4. When we compare equation (4) with equation (1) we notice that dM/dt occurs implicitly in the first term of both equations, and that the use of $-u\,dM/dt$ as the second term in (1) imposes a restriction on the nature of the problems to which this equation applies; namely, if (1) is true for a problem, then all the time rates of change of mass that may be taking place simultaneously must together be expressible by a single symbol dM/dt having the same meaning and value as the dM/dt implicit in the first term of the equation. It should be evident from what precedes that simple problems on changing mass cannot all be limited in this way.

5. The examples which I have given were primarily intended to illustrate the fact that rate of change of momentum cannot always be expressed by a single differential coefficient. They also show that when a variety of physical or chemical changes are taking place simultaneously the rate of change of mass is not necessarily expressible only in the form dM/dt.

6. The simplest form of equation of motion that includes all the problems I have suggested appears to be

$$d(MV)/dt + \Sigma(M'V') = F, \qquad (5)$$

where M denotes the mass of the body and V its velocity at time t, and M' denotes the rate at which any particular kind of matter is being ejected and V' the component velocity of this matter in the direction of F, the sign of any term under the Σ being changed if the matter is being absorbed.

It will be noticed that we get (3) from (5) by putting $M' = m'$ and $V' = V$; and that to get (4) from (5) we have to take the sum of three terms:

$-(m' - m)v$ for the oxygen taken into combination,

$m''(V - v')$ for the ashes ejected, and

$(m' - m'')V$ for the smoke and gases ejected.

I am not prepared to claim any generality for equation (5) beyond its applicability to the kind of problem indicated. As to the value or importance of the existence of such a general equation I take it to be *nil*. I should regard the ability to quote and use it as a poor substitute for such a grasp of the principle of change of momentum as will enable a student to apply it for himself.

A. S. R.

1407. Has the reader ever noticed how often an eight is a singleton? Perhaps some expert mathematician can say whether this has anything to do with the fact that it is the middle card of the thirteen. Probably the cause of this curious happening is due to incidents of play affecting the inner pattern of the pack in a way which defeats normal probabilities. We assume that any one named card in a *new* pack is as likely to be a singleton as any other card.—*Sunday Times*, July 7, 1940.

1408. " Lance-Jack " writes:

A fellow in my room says that one horse-power will lift one pound 33,000 feet in one minute, or 33,000 lb. one foot in one minute.

If both statements are correct I lose a packet of smokes.

Answer: Fear not.

One horse-power raises 550 lb. one foot in one second.—*Daily Mirror.* [Per Mr. J. E. Blamey.]

1409. "Not locus if you will but envelope,
Paths of light not atoms of good form;
Such tangent praise, less crashing, not less warm,
May gain more intimacy for less hope."

—William Empson, " Letter V ", printed in *The Century's Poetry*, p. 450. [Per Mr. F. W. Kellaway.]

1410. I do believe in simplicity. When the mathematician would solve a difficult problem he first frees the equation from all encumbrances, and reduces it to its simplest terms. So simplify the problem of life, distinguish the necessary and the real.—D. H. Thoreau, quoted in the *Cambridge History of American Literature*, II. [Per Mr. F. J. Wood.]

MATHEMATICAL NOTES

1611. *Maclaurin's formula.*

It is the main purpose of this note to show that the usual drill in expanding functions in power series can be put on a sound foundation without any reference to the theory of convergence of infinite series.

We begin with Maclaurin's theorem in the form :

If $f^{(n)}(0)$ *exists, then*

$$f(x)=f(0)+xf'(0)+(x^2/2!)f''(0)+\ldots+(x^n/n!)f^{(n)}(0)+\alpha x^n,$$

where $\alpha\to 0$ *when* $x\to 0$.

It may be noted that this implies that if

$$f(x)=S_n+R_n,$$

where S_n is the sum of the first n terms, then

$$R_n=\alpha x^{n-1},$$

where $\alpha\to 0$ as $x\to 0$, provided only that $f^{(n-1)}(0)$ exists. The usual forms of R_n, due to Lagrange and Cauchy, require the existence of $f^{(n)}(x)$ and require it over an interval.

The theorem appears to be due to W. H. Young. It is proved in de la Vallée Poussin, *Cours d'analyse*, I, p. 77 (5th edition) and in Hardy, *Pure Mathematics*, p. 289 (7th edition). The steps of an alternative proof are indicated below under (1) – (4). See Phillips, *Analysis*, p. 110 (2nd edition), or Chaundy, *Differential Calculus*, p. 112.

(1) *L'Hospital's first rule.*

Let $F(0)=G(0)=0$, and let $F'(0)$, $G'(0)$ exist, $G'(0)\neq 0$. Then

$$\lim_{x\to 0}\frac{F(x)}{G(x)}=\frac{F'(0)}{G'(0)}.$$

(2) *L'Hospital's second rule.*

Let $F(0)=G(0)=0$. Let $F(x)$, $G(x)$ be continuous at $x=0$ and differentiable in the neighbourhood of $x=0$ with the possible exception of the point $x=0$ itself. Then

$$\lim_{x\to 0}\frac{F(x)}{G(x)}=\lim_{x\to 0}\frac{F'(x)}{G'(x)},$$

provided that this latter limit exists.

(3) *Extension of L'Hospital's rule.*

Let $F(x)$, $G(x)$ and their first $(n-1)$ derived functions all vanish at $x=0$. Let $F^{(n)}(0)$, $G^{(n)}(0)$ exist, $G^{(n)}(0)\neq 0$. Then

$$\lim_{x\to 0}\frac{F(x)}{G(x)}=\frac{F^{(n)}(0)}{G^{(n)}(0)}.$$

This follows by $n-1$ applications of L'Hospital's second rule and a final application of the first rule.

(4) *Maclaurin's formula.*

This follows at once, in the above form, by applying (3) to the functions

$$F(x) = f(x) - f(0) - xf'(0) \ldots - (x^n/n!)f^{(n)}(0),$$
$$G(x) = x^n.$$

Corollary. If $f^{(n)}(0)$ exists for all n, then Maclaurin's formula is true for all n, where $\alpha \to 0$ when $x \to 0$ for every fixed value of n.

Maclaurin's formula can now be applied to obtain the expansions of the usual functions for which $f^{(n)}(0)$ exists. Besides the usual examples, others can be included, such as

$$e^{-1/x} = 0 + 0 \,.\, x + 0 \,.\, x^2 + \ldots + 0 \,.\, x^n + \alpha x^n, \quad (x > 0),$$

or

$$e^x + e^{-1/x} = 1 + x + (x^2/2!) + \ldots + (x^n/n!) + \alpha x^n, \quad (x > 0).$$

(5) *Differentiation and integration of Maclaurin formulae.*

If $f^{(n)}(0)$ exists, then besides the Maclaurin formula for $f(x)$, the following formulae hold good also :

$$f'(x) = f'(0) + xf''(0) + \ldots + \{x^{n-1}/(n-1)!\}f^{(n)}(0) + \beta x^{n-1},$$
$$\int_0^x f(t)dt = xf(0) + (x^2/2!)f'(0) + \ldots + \{x^{n+1}/(n+1)!\}f^{(n)}(0) + \gamma x^{n+1},$$

where $\beta \to 0$, $\gamma \to 0$ as $x \to 0$.

The first of these holds good because the $(n-1)$-th derivative of $f'(x)$ at $x = 0$ exists, for it is $f^{(n)}(0)$; the second because the $(n+1)$-th derivative of $\int_0^x f(t)dt$ at $x = 0$ exists, for it is $f^{(n)}(0)$.

Hence a Maclaurin formula can be differentiated or integrated "term by term" ; consequently, the Maclaurin formula for $f(x)$ can be found by integrating that for $f'(x)$ or by differentiating that for $\int_0^x f(t)dt$, if either of these can be found more easily.

(6) The next step is to prove that any formula of the type

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \alpha x^n,$$

where $c_0, c_1 \ldots c_n$ are independent of x, and $\alpha \to 0$ when $x \to 0$, is unique.

It will follow that if a formula of this type is obtained by *any* means (for example, algebraically), and if $f^{(n)}(0)$ is known to exist, then the formula obtained must be the Maclaurin formula for $f(x)$.

Consequently, algebraic operations can be applied to Maclaurin formulae. Thus, suppose $u(x)$, $v(x)$ are functions of x such that $u^{(n)}(0)$, $v^{(n)}(0)$ exist, and for which the Maclaurin formulae are known. Then by algebraic operations we can find formulae of the above type for the sum, product and quotient of u and v, that is, for the functions

$$f(x) = u + v,\ f(x) = uv,\ f(x) = u/v,$$

where we must suppose $v(0) \neq 0$ in the last case. But in all these cases, $f^{(n)}(0)$ is known to exist. Hence the formulae obtained by algebraic methods must be Maclaurin formulae.

Little more discussion is necessary to show that algebraic operations are justifiable when they can be used to find the Maclaurin formula for a function of a function, or to " invert " a Maclaurin formula.

(7) Finally, *asymptotic series* can be introduced.

We write

$$f(x) \sim c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots ,$$

and call this the asymptotic expansion of $f(x)$ at $x=0$, if the coefficients $c_0, c_1, c_2 \ldots$ are independent of x and if, for every fixed value of n,

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \alpha x^n,$$

where $\alpha \to 0$ when $x \to 0$.

It follows that, if $f^{(n)}(0)$ exists for all n, the Maclaurin *series* is always asymptotic to $f(x)$ at $x=0$. [The series may hold good for $x>0$ only, as in the case of the series for $\exp(-1/x)$; or for $x<0$ only, as in the case of $\exp(1/x)$: or for real values of x only, as in the case of $\exp(-1/x^2)$].

As regards the convergence of Maclaurin's series, it may be pointed out that for the convergence of the series, $R_n(x)$ must tend to a limit when $n \to \infty$, since $R_n(x) = f(x) - S_n(x)$. If the limit is zero, the series converges to the sum $f(x)$; if the limit is not zero, the series converges, but not to $f(x)$. Incidentally, this brings out the fact that $R_n(x)$ is not " the remainder after n terms " of the series.

F. Bowman.

1612. *An example in differentiation.*

Prove that

$$\left(\frac{d}{dx}\right)^{2n} (1-x^2)^{n-\frac{1}{2}} = \left\{\frac{(2n)!}{2^n \, . \, n!}\right\}^2 \cdot \frac{(-)^n}{(1-x^2)^{n+\frac{1}{2}}};$$

and, more generally,

$$\left(\frac{d}{dx}\right)^{2n} (ax^2+2bx+c)^{n-\frac{1}{2}} = \left\{\frac{(2n)!}{2^n \, . \, n!}\right\}^2 \cdot \frac{(ac-b^2)^n}{(ax^2+2bx+c)^{n+\frac{1}{2}}}.$$

F. Bowman.

1613. *On a geometrical construction.*

My attention was drawn recently to the problem of bisecting a given line by the use of *compasses* only. The solution may be fairly well known, but since the problem was new to me, some *Gazette* readers may be interested in the following two solutions.

It is clear that the solution must be found by drawing a series of circles. The first three are fairly obvious. Let AB be the given line. Draw the circles S_1, S_2 with centres A and B and radii AB. If F is one of their two points of intersection, S_3 is the circle centre F and radius FA (or FB). What I shall call the " standard "

solution is now obtained by drawing the circles S_4, S_5. If C is the point where S_3 cuts S_2, S_4 is the circle centre C, radius CB. If S_4 cuts S_2 at G, S_5 has centre G and radius GA. S_5 cuts S_1 at H, and H is equidistant from A and M (the mid-point of AB). The proof is quite easy. Since G is the point on AB produced such that $AB = BG$, we easily prove that $\cos HAG = 1/4$. If $AB = 2a$, $AH = 2a$, and it is easy to show that $HM = 2a$.

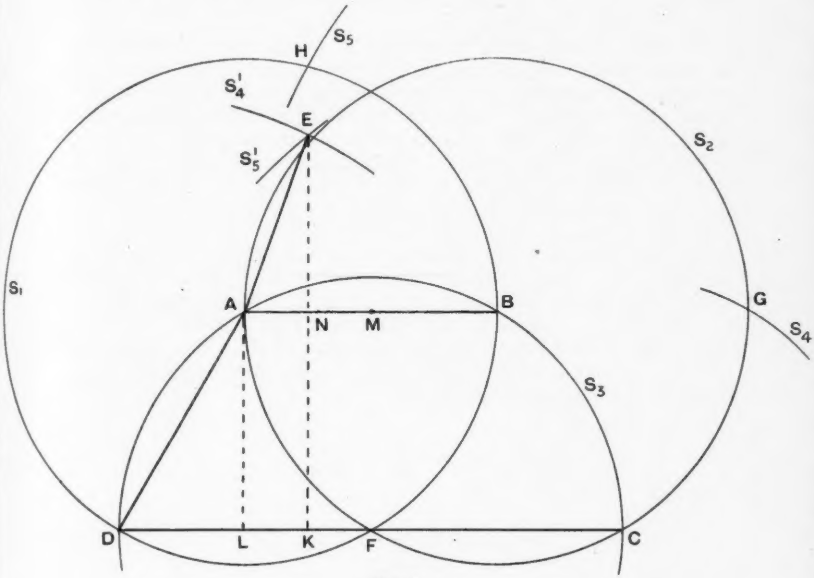


FIG. 1.

There is, however, a second solution which is rather interesting. Actually it is the solution which I found *first*, by approaching the problem experimentally. I drew the circles S_1, S_2, S_3, as in the "standard" solution. S_4' is the circle centre D, radius DB, and S_5' is the circle centre C and radius CD. S_4' and S_5' intersect at a point E which is equidistant from A and M. To prove this, draw AL and EK perpendicular to DC. Since $DABC$ is half of a regular hexagon $DB(=DE) = 2\sqrt{3}a$; $CE(=CD) = 4a$. If $DK = x$, $EK = y$, then $CK = 4a - x$, and we have

$$x^2 + y^2 = 12a^2, \quad (4a - x)^2 + y^2 = 16a^2,$$

and by subtraction we get $x = 3a/2$. Since DAF is an equilateral triangle, $DL = \frac{1}{2}DF = a$, hence $LK = AN = \frac{1}{2}a$. It follows that $EA = EM$.

This second solution is interesting, because, by drawing fairly

obvious circles, we find a point equidistant from A and M, but its distance from A is h, where h is given by the relation

$$4h^2 = (13 - 3\sqrt{13})a^2.$$

In the "standard" solution we find a point H whose distance from A is equal to the length of the given line AB. In the second solution we find a point E whose distance from A is an irrational number, yet the point E is just as easily found by drawing a series of circles as is the point H.

As I have already said, the point E is the one which I found first on approaching the problem with no knowledge of the method of solution. On showing it to my colleague, Dr. R. A. Newing, he found the "standard" solution, which is probably the solution best known to those who may be familiar with the problem.

The solution of the problem of bisecting a given straight line by compasses only was shown to Dr. T. Estermann, who suggested that the problem of finding the centre of a given circle, by the use of compasses only, might also be solved in a similar way. Dr. Estermann's solution is as follows. Take any point A on the circle S. With any radius a, greater than the radius of the given circle, strike two arcs, S_1, S_2, cutting S at B and C. With centres B and C and radius a, complete the rhombus $ABDC$. With centre D, radius DA, draw circle S_3 cutting S_1 and S_2 at E and F. Then the two circles with centres E and F and radius EA intersect at the centre O of the given circle (Fig. 2).

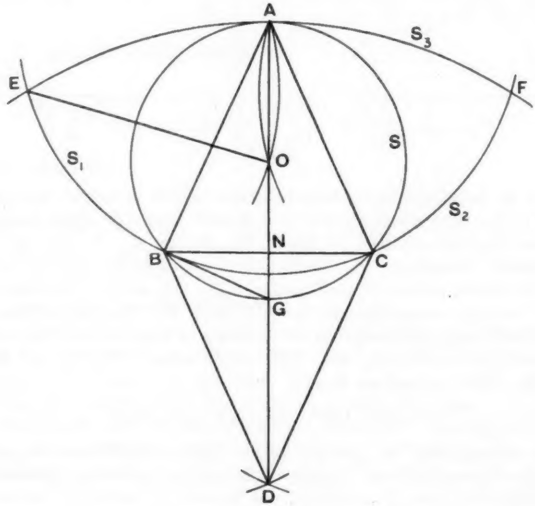


FIG. 2.

The proof is quite elementary. Since the triangles ABN and ABG are similar, if $AD=2b$, from $AB:AG=AN:AB$, we get

$$a^2=2br, \qquad \text{.................................(1)}$$

since, if r is the radius of S, $AG=2r$.

Since $EA=EO$, $DA=DE$, the isosceles triangles EAO and EDA are similar, hence $AO:EA=EA:AD$, giving

$$a^2=2b\,.\,AO. \qquad \text{..............................(2)}$$

Hence from (1) and (2), $AO=r$. E. G. PHILLIPS.

1614. *The Terry "Anglepoise" Lamp.*

This common and intriguing piece of apparatus depends for its working on simple principles, the triangle of forces, resolution into oblique components (not often illustrated in real life), the Law of the Lever, Hooke's Law; it is therefore a useful illustration for revision and discussion.

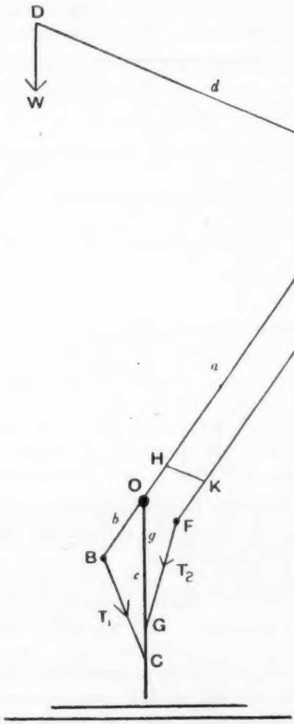


FIG. 1.

The lamp is shown diagrammatically in Fig. 1. OGC is a fixed vertical support; BC and FG are spiral springs; the bulb and reflector are supported by universal joints at the end of the light arm EAD in such a way that their C.G. always lies in the line of the arm at D. $AHOB$, EKF, and KH are smoothly jointed light rods. It is found that the lamp will remain in equilibrium in any position.

The mechanics of the apparatus is best elucidated with the aid of two Lemmas.

Lemma I. BC is a spring with one end fixed at C, vertically below the fixed point of suspension O of a lever AOB. *If the spring BC be so wound that its tension is proportional to its length* (*i.e.* its "natural" length is zero), then a suitable weight W at A will be in equilibrium in any position of AOB.

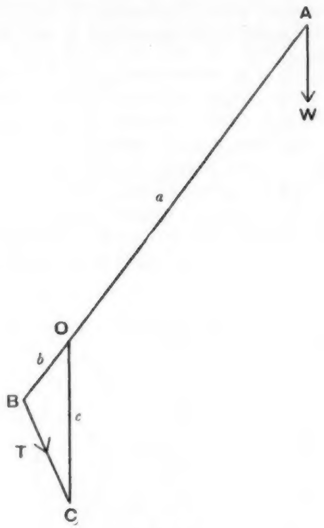


FIG. 2.

Proof. Let the tension T in BC be resolved into a component F along BO and a component W_1 along the downward vertical at B. BOC is a force triangle for this resolution.

Hence $W_1/c = T/BC = \lambda_1$, say (Fig. 2); and so $W_1 = \lambda_1 . c =$ constant, and F has no moment about O. Hence, provided $W_1 . b = W . a$, there is equilibrium in any position. The required condition may be written $\lambda_1 = W . a/bc$.

Lemma II. If in Fig. 1 $AEKH$ is a smoothly hinged parallelogram, and provided the weight of HK can be neglected, the forces at F, O, required for equilibrium are independent of the position of HK, as long as it is parallel to AE.

This is fairly evident, but is easily proved by Virtual Work, the displacements of everything except HK itself being unaffected by the position of HK.

We shall neglect the weight of the other rods, though this is not necessary, and they can easily be allowed for.

Consider now the lamp itself. The tension $T_1 = \lambda_1 . BC$ can by Lemma I be replaced by a vertical force at B, $W_1 = \lambda_1 . c$, and a force along BO which has no moment about O.

In the same way $T_2 = \lambda_2 . FG$ can be replaced by a vertical force at F, $W_2 = \lambda_2 . g$, and a force along FO which has no moment about O.

Resolve W, W_1, W_2 into components along AO (the a-component, say) and DA (the d-component).

Then if $W_1 . b = W . a$, and $W_2 . e = W . d$, and OF is parallel to AE,

the d-component of W_1 balances the d-component of W;
the a-component of W_1 has no moment about O;
the a-component of W_2 balances the a-component of W;
the d-component of W_2 has no moment about O.

Hence the apparatus will balance in all positions, subject to these conditions, which reduce to

$$\lambda_1 = W . a/bc, \quad \lambda_2 = W . d/eg.$$

The last step can also be seen without the use of oblique components as follows:

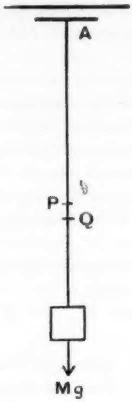
Complete the parallelogram $DAOL$. Then W at D can be replaced by W at A, W at L, and $-W$ at O, which last force has no effect. If $W_1 . b = W . a$, W_1 balances W at A. If $W_2 . e = W . d$, W_2 balances W at L.

H. Martyn Cundy.

1615. *The oscillations of a mass suspended on a heavy elastic spring.*

It should be known to all Physics candidates for the Higher Certificate that in the above well-known experiment the proper correction for the mass of the spring in calculating g from the periodic time is to add $\frac{1}{3}$ the mass of the spring to the suspended mass. The more mathematically-minded candidate may inquire how this is obtained, and in addition ask awkward questions about the equilibrium configuration of the spring; as no school textbook to my knowledge deals with the matter, and the mathematics required is not beyond the powers of a good boy, I append here an account which has no pretensions to originality, and can probably be replaced by something much simpler, but which may assist a busy member in distress.

Equilibrium. Let A be the point of suspension, a the unstretched length, and l the stretched length of the spring. Let M be the suspended mass, m the mass of the spring, and ρ_0 its mass per unit unstretched length. Let the unstretched length of the part AP be x_0, and its equilibrium stretched length x_e.



For a length $PQ(\delta x_0)$ we have

$$\delta T_e \doteqdot -\rho_0 \, . \, g \, . \, \delta x_0, \qquad \text{(i)}$$

where the equilibrium tension T_e is given by Hooke's Law *in its differential form,*

$$T_e = k \, . \left\{ \frac{dx_e}{dx_0} - 1 \right\}.$$

From (i) we have

$$dT_e/dx_0 = -\rho_0 g,$$

and therefore $$T_e = (M+m)g - \rho_0 g x_0,$$

since $$T = (M+m)g \text{ when } x_0 = 0.$$

Thus $$dx_e/dx_0 = 1 + T_e/k = 1 + (M+m)g/k - \rho_0 \, . \, g \, . \, x_0/k, \qquad \text{(ii)}$$

and hence $$x_e = x_0 + (M+m)gx_0/k - \rho_0 \, . \, g \, . \, x_0^2/2k,$$

since $$x_e = 0 \text{ when } x_0 = 0.$$

This gives the equilibrium configuration of the spring.

Further, since $x_e = l$ when $x_0 = a$,

$$\begin{aligned} l &= a + (M+m)ga/k - \rho_0 g a^2/2k \\ &= a + Mga/k + \tfrac{1}{2}mga/k, \end{aligned}$$

or $$(l-a)/a = (M + \tfrac{1}{2}m)g/k,$$

i.e. the correction in equilibrium to the usual formula is *one-half the mass of the spring.* This correction does not usually have to be made in a physical experiment, since " k " is found from *differences* of extension caused by different loads M.

Vibration. Suppose AP is extended a further distance $\xi(x_0)$, and let B descend a distance $X = \xi(a)$.

Then the depth of P below $A = x_e + \xi$,

and that of Q below A $\quad = x_e + \delta x_e + \xi + \dfrac{\partial \xi}{\partial x_e} \cdot \delta x_e$.

Hence
$$T = k \cdot \left\{ \delta x_e + \frac{\partial \xi}{\partial x_e} \cdot \delta x_e - \delta x_0 \right\} \Big/ \delta x_0$$
$$= ks \cdot \frac{\partial \xi}{\partial x_e} + k(s-1), \text{ writing } \frac{\partial x_e}{\partial x_0} = s,$$
$$= k \cdot \partial \xi / \partial x_0 + k(s-1). \qquad \text{(iii)}$$

Also the equation of motion for PQ gives
$$\delta T + \rho_0 g \, \delta x_0 = \rho_0 \delta x_0 \frac{\partial^2 \xi}{\partial t^2}.$$

Thus
$$\frac{\partial^2 \xi}{\partial t^2} = g + \frac{1}{\rho_0} \frac{\partial T}{\partial x_0}$$
$$= g + \frac{k}{\rho_0} \left\{ \frac{\partial^2 \xi}{\partial x_0^2} + \frac{\partial^2 x_e}{\partial x_0^2} \right\}, \text{ from (iii).}$$

Now
$$\frac{\partial^2 x_e}{\partial x_0^2} = \frac{\partial s}{\partial x_0} = -\rho_0 \cdot g/k, \text{ from (ii).}$$

Hence we obtain the final partial differential equation for the vibration
$$\frac{\partial^2 \xi}{\partial t^2} = \frac{k}{\rho_0} \cdot \frac{\partial^2 \xi}{\partial x_0^2}, \qquad \text{(iv)}$$

which is the usual " Equation of Sound ".

For the motion of the mass M we have
$$M\ddot{X} = Mg - T(l + X), \qquad \text{(v)}$$

and from (iii),
$$T(l+X) = k \frac{\partial \xi}{\partial x_0} + k(s-1) \text{ at } x_e = l$$
$$= k \frac{\partial X}{\partial x_0} + T_e = k \frac{\partial X}{\partial x_0} + Mg.$$

Thus
$$M\ddot{X} + k \frac{\partial X}{\partial x_0} = 0. \qquad \text{(vi)}$$

Equation (iv) has solutions of the form $\xi = A \cdot \cos nt \cdot \sin nx_0/c$, where $c^2 = k/\rho_0$. We then have $X = A \cdot \cos nt \cdot \sin na/c$, and (vi) gives
$$-MAn^2 \cos nt \sin na/c + kAn/c \cdot \cos nt \cos na/c = 0\,;$$

and so
$$n \tan (na/c) = k/Mc.$$

Let $\theta = na/c$. Then $\theta \cdot \tan \theta = ka/Mc^2 = a\rho_0/M = m/M$.

The higher roots approach $r\pi$, giving the ordinary harmonics of the spring, but the smallest root is given approximately by

$$\theta(\theta+\tfrac{1}{3}\theta^3+\dots)=m/M,$$

whence
$$\theta^4+3\theta^2-3(m/M)=0,$$

and
$$\theta^2=(m/M)-\tfrac{1}{3}\theta^4$$
$$\doteqdot(m/M)-\tfrac{1}{3}(m/M)^2, \text{ for small } (m/M).$$

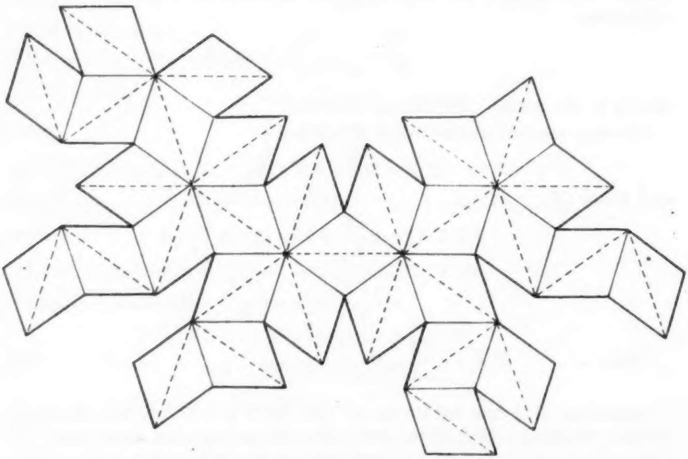
Hence the period of vibration

$$=2\pi/n=2\pi a/c\theta$$
$$\doteqdot\frac{2\pi a}{c}\cdot\sqrt{\left(\frac{M}{m}\right)}\cdot\left(1-\frac{1}{3}\frac{m}{M}\right)^{-\frac{1}{2}}$$
$$=2\pi a\sqrt{\left\{\frac{M\rho_0}{mk}\left(1+\frac{1}{3}\frac{m}{M}\right)\right\}}$$
$$=2\pi\sqrt{\{a(M+\tfrac{1}{3}m)/k\}},$$

which shows the correcting addition of one-third the mass of the spring. H. MARTYN CUNDY.

1616. *On Note* 1539.

I was much interested in this note, having tried myself to make a model of the great stellated dodecahedron there illustrated. It is



NET FOR THE GREAT STELLATED DODECAHEDRON.

The solid lines should be scored on the front, the dotted lines on the back and folded forward. The front then becomes the inside.

an elegant solid, pleasing to the eye, and its very existence in cardboard delights the uninitiated. I fear, not having seen the note

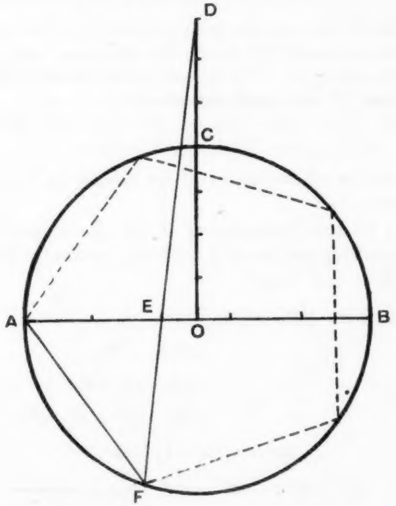
with its practical advice of using airplane cement, I struggled with an arrangement of slotted tabs and Gripfix, and ended by having to cover the most unsightly corners with small paper circles. However, I succeeded in cutting it out of *one* piece of cardboard, which I think is better than Mr. Cadwell's use of 11 pieces. As I did not at first believe this possible, I append the net used : it may stimulate others to construct this delightful solid. (For simplicity, tabs are omitted.)

Incidentally I remember with pleasure even now the sight of a complete set of models of the 59 icosahedra obtainable by producing the planes of the regular icosahedron (of course, not necessarily vertically or facially regular). They were made under the instructions of Dr. H. S. M. Coxeter. See H. S. M. Coxeter, P. du Val, H. T. Flather and J. F. Petrie, *The Fifty-nine Icosahedra* (University of Toronto Press, 1938).

H. MARTYN CUNDY.

1617. *Approximate method of inscribing a regular polygon in a circle.*

The following " rough-and-ready " method of inscribing a polygon in a given circle was shown me by a draughtsman friend, who believed it to be a good approximation for polygons of any number of sides. It is easy to see that it is not very accurate for polygons with n, the number of sides, large, but I was surprised to find its extraordinary accuracy for values of n from 3 to 12, for which cases, of course, it is most likely to be used.



The construction is as follows ;

Let A be the position of one of the vertices of the required polygon. Draw the diameter AOB of the circle, and a perpendicular radius OC. Divide OC into 4 equal parts, and from OC produced cut off 3 more of these parts, so making $OD = 7OC/4$. Divide AB into n parts, the number of sides of the polygon required, and let AE contain 2 of these parts. Join DE and produce it to meet the circle in F. Then AF is the first side of the required polygon.

The accuracy of the construction can be gauged from the accompanying table, which gives the values of the angle AOF calculated from the details of the construction, the true values, and the error per cent.

n	$\angle AOF$ Constructed.	$\angle AOF$ True.	Percentage error.
3	119° 53·9′	120°	– ·083
4	90°	90°	0
5	72° 1·1′	72°	·025
6	60° 6·1′	60°	·17
7	51° 38·3′	51° 25·7′	·41
8	45° 19·2′	45°	·74
9	40° 25′	40°	1·0
10	36° 30′	36°	1·4
11	33° 17′	32° 43·6′	1·7
12	30° 37′	30°	2·0
20	18° 45′	18°	4·25

For large values of n, if the true position of F is joined to E and FE is produced to meet OC in D, the limiting value of OD/OC is easily shown to be $\pi/2$. The above table shows how nearly this ratio approaches 7/4 for small values of n.

H. MARTYN CUNDY.

1618. *The radius of curvature of an ellipse in terms of the focal distances.*

Let R and r be the distances of P, on the ellipse, from the foci S and s, both making angles of ϕ with the normal. Then $R + r = 2a$.

Also let $\sqrt{(Rr)} = p$. Then

$$\begin{aligned} R^2 + r^2 - 2Rr \cos 2\phi &= Ss^2 \\ &= 4(a^2 - b^2) \\ &= (R + r)^2 - 4b^2 \\ &= R^2 + 2p^2 + r^2 - 4b^2. \end{aligned}$$

Hence $$2p^2(\cos 2\phi + 1) = 4b^2.$$

Thus $$4p^2 \cos^2 \phi = 4b^2 \quad \text{and} \quad \cos \phi = b/p.$$

Inwards along the normal, measure a distance z, reaching to C.

Then $\quad CS^2 + Cs^2 = R^2 + z^2 - 2Rz\cos\phi + r^2 + z^2 - 2rz\cos\phi$

$$= 2z^2 + (R^2 + r^2) - 4az\cos\phi$$

$$= 2z^2 + 4a^2 - 2p^2 - 4abzp^{-1}.$$

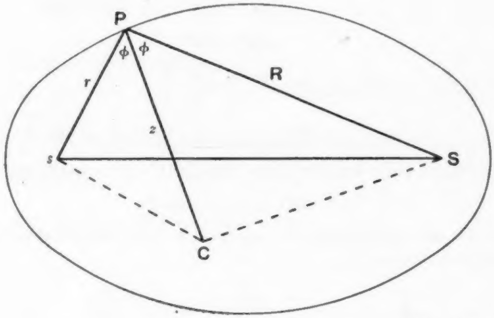
If C is the centre of curvature, two consecutive normals cut at C.

Thus $\quad \dfrac{dz}{dp} = 0, \quad$ and $\quad \dfrac{d}{dp}(CS^2 + Cs^2) = 0.$

Hence $\quad -4p + 4abzp^{-2} = 0,$

and so z, the radius of curvature, $= p^3/ab$

$$= (Rr)^{3/2}/ab.$$



W. Hope-Jones.

1619. *The addition formulae.*

The following elementary geometrical proofs of the expansions for $\sin(A \pm B)$ and $\cos(A \pm B)$, depend only upon the $\frac{1}{2}ab\sin C$ formula for the area of a triangle, and the cosine rule.

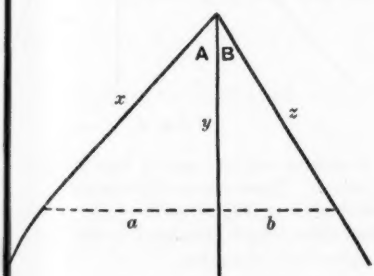


Fig. 1.

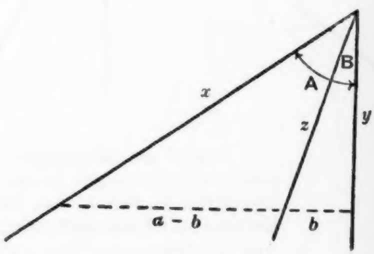


Fig. 2.

A and B are two angles having a common vertex, and a line is drawn at right angles to the common arm y. Then where the upper

sign refers to Fig. 1 (sum) and the lower sign to Fig. 2 (difference), the area of the triangle contained by the sides x and z, is equal to the sum or difference of the areas of the other two triangles.

Hence

$$\tfrac{1}{2}xz \sin (A \pm B) = \tfrac{1}{2}xy \sin A \pm \tfrac{1}{2} yz \sin B.$$

Thus
$$\sin (A \pm B) = \sin A \,.\, \frac{y}{z} \pm \frac{y}{x} \,.\, \sin B,$$

i.e.
$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

Using the cosine rule in the triangle contained by the sides x and z,

$$\begin{aligned} 2xz \cos (A \pm B) &= x^2 + z^2 - (a \pm b)^2, \\ &= (x^2 - a^2) + (z^2 - b^2) \mp 2ab \\ &= 2y^2 \mp 2ab. \end{aligned}$$

Thus
$$\cos (A \pm B) = \frac{y}{x} \,.\, \frac{y}{z} \mp \frac{a}{x} \,.\, \frac{b}{z},$$

i.e.
$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

If in addition it is known that $\cos (360° - \theta) = \cos \theta$ and

$$\sin (360° - \theta) = -\sin \theta,$$

these proofs are applicable to angles of any magnitude, and it is here that their value lies.

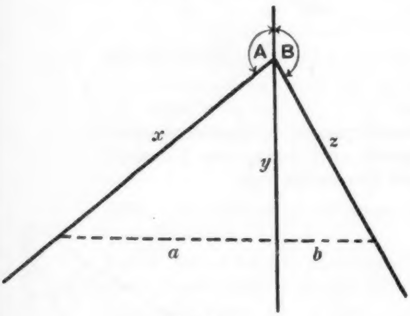


FIG. 3.

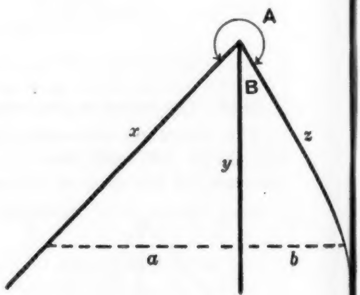


FIG. 4.

A and B are two angles having a common vertex, and a line is drawn at right angles to the common arm y. Then where the upper sign refers to Fig. 3 (sum) and the lower sign to Fig. 4 (difference), the area of the triangle contained by the sides x and z is equal to the sum or difference of the areas of the other two triangles.

Hence $\quad \tfrac{1}{2}xz \sin (360° - \overline{A \pm B}) = \tfrac{1}{2}xy(\pm \sin A) + \tfrac{1}{2}yz \sin B.$

Thus
$$-\sin (A \pm B) = (\pm \sin A)\frac{y}{z} + \frac{y}{x} \sin B,$$

i.e. $$-\sin(A \pm B) = -\sin A \cos B \mp \cos A \sin B,$$

or $$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

Using the cosine rule in the triangle contained by the sides x and z,

$$\begin{aligned} 2xz \cos(360^\circ - \overline{A \pm B}) &= x^2 + z^2 - (a+b)^2, \\ &= (x^2 - a^2) + (z^2 - b^2) - 2ab, \\ &= 2y^2 - 2ab. \end{aligned}$$

Thus $$\cos(A \pm B) = \frac{y}{x} \cdot \frac{y}{z} - \frac{a}{x} \cdot \frac{b}{z},$$

i.e. $$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

R. J. GILLINGS.

1620. *A note on differentials.*

Let $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ be single-valued differentiable functions of the independent variables x, y.

Theorem. The necessary and sufficient condition that $x\,dy - \xi\,d\eta$ is an exact differential, say, $d[\phi(x, y)]$ is that $\dfrac{\partial(x, y)}{\partial(\xi, \eta)} = 1$.

Proof. (a) The condition is necessary: for if

$$x\,dy - \xi\,d\eta = d\phi,$$

then $$x\frac{\partial y}{\partial \xi} = \frac{\partial \phi}{\partial \xi} \text{ and } x\frac{\partial y}{\partial \eta} = \frac{\partial \phi}{\partial \eta} + \xi:$$

hence $$\frac{\partial}{\partial \eta}\left(x\frac{\partial y}{\partial \xi}\right) = \frac{\partial}{\partial \xi}\left(x\frac{\partial y}{\partial \eta} - \xi\right). \quad \text{..................(i)}$$

Therefore $$\frac{\partial x}{\partial \eta} \cdot \frac{\partial y}{\partial \xi} = \frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} - 1.$$

(b) The condition is sufficient: for if

$$\frac{\partial(x, y)}{\partial(\xi, \eta)} = 1,$$

we deduce equation (i), and hence

$$\left(x\frac{\partial y}{\partial \xi}\right) d\xi + \left(x\frac{\partial y}{\partial \eta} - \xi\right) d\eta$$

is exact.

If C is any simple closed curve in the (x, y) plane, enclosing an area S, and if Γ is the corresponding closed curve in the (ξ, η) plane, of area Σ,

$$\int_{(C)} x\,dy - \int_{(\Gamma)} \xi\,d\eta = \int_{(C)} d\phi = 0;$$

and hence $$S = \Sigma.$$

This follows at once, alternatively, by using the unit value of the Jacobian above, since

$$\iint_{(S)} dx\,dy = \iint_{(\Sigma)} \frac{\partial(x, y)}{\partial(\xi, \eta)}\,d\xi\,d\eta = \iint_{(\Sigma)} d\xi\,d\eta.$$

J. H. PEARCE.

1621. *Lottery chances.*

In a certain lottery n tickets win prizes and n^2 are blanks. If a man buys n tickets, what is his chance of a prize?

The chance that none of the n tickets bought wins a prize is

$$\binom{n^2}{n} \Big/ \binom{n^2+n}{n} = \{(n^2)!\}^2/(n^2-n)!\,(n^2+n)!$$

$$\sim n^{4n^2+2}/(n^2-n)^{n^2-n} \,.\, (n^2+n)^{n^2+n} \,.\, n(n^2-1)^{\frac{1}{2}}$$

by Stirling's formula,

$$=\{n/(n^2-1)^{\frac{1}{2}}\}(1-1/n)^n(1+1/n)^{-n}(1-1/n^2)^{-n^2}$$

$$\to 1/e.$$

Thus, if n is large, the chance of a prize is nearly $1-1/e$, *i.e.* nearly 17/27.

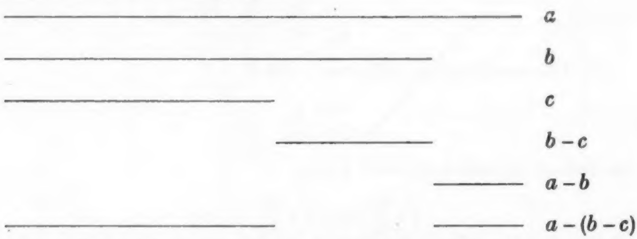
For instance, if there are 100 prizes and 10,000 blank tickets, the holder of 100 tickets wins a prize as often as 17 in 27 times, and yet to be certain of a prize no less than 10,001 tickets must be bought.

R. L. G.

1622. "*A minus and a minus makes a plus.*"

"Seeing is believing", and a visual aid certainly does help some pupils, even if not all. This diagram may be useful in supplementing whatever method is usually used to deal with this point.

Take any three unequal lengths a, b, c, and the rest follows from an examination of the diagram.



Naturally the diagram without a word of explanation from the teacher would not be of much use; but it can be seen that, in the last line, the c has, as it were, re-appeared. By inspection

$$a-(b-c)=a-b+c.$$

This may well be followed either by each member of the class choosing his own numerical example and working it out by two different routes, for example,

$$10-(7-2)=10-5=5,$$

or
$$10-(7-2)=10-7+2$$
$$=12-7=5,$$

or if preferred one could show diagrammatically that

$$a-2(b-c)=a-2b+2c,$$
$$a-(b-2c)=a-b+2c,$$
$$a-2(b-2c)=a-2b+4c.$$

These too can be followed by numerical examples.

The following mnemonic amuses older pupils—particularly if it is mentioned that it has actually been given seriously as a teaching method. If we wrote two minus signs in succession, thus – –, they might get run into each other and mistaken for just one —. Se we turn one on end and write them like this : + .

C. DUDLEY LANGFORD.

1623. *Simple tests for divisibility by* 7 *and* 13.

In much elementary arithmetical work we may need to find the factors of a given number. The rules for testing divisibility by 2, 3, 4, 5, 6, 8, 9, 10, 11 and 12 are well known, though it is remarkable how many school children maintain that they have never been shown them. The tests for other numbers are not so simple, but those for 7 and 13 can be combined in the following manner :

(i) separate the given number into groups of three digits by commas in the usual way ;

(ii) add alternate groups separately ;

(iii) deduct the smaller from the larger total ;

(iv) if the result still has more than three digits, repeat the process.

Then if the resulting three-digit number is divisible by 7 or 13, so is the original number.

This is probably the best form of the rule for most pupils since there is not much chance of making a mistake, but one can proceed, if one wishes, with the three-digit number in this way :

(i) Divide by 50, add the quotient to the remainder ; if the result is divisible by 7, so is the original.

(ii) Divide by 40, add the quotient to the remainder : if the result is divisible by 13, so is the original.

Examples. (i) Test 2, 634, 224, 681, 998.

$$\begin{array}{r} 2 \\ 224 \\ \underline{998} \\ 1224 \end{array} \qquad \begin{array}{r} 634 \\ \underline{681} \\ 1315 \end{array} \qquad \begin{array}{r} 1315 \\ \underline{1224} \\ 91 \end{array}$$

Now 91 is divisible by 7 and by 13, and hence so is the original.

(ii) Test 28, 976, 471, 501, 978.

$$\begin{array}{r} 28 \\ 471 \\ \underline{978} \\ 1477 \end{array} \qquad \begin{array}{r} 976 \\ \underline{501} \\ 1477 \end{array}$$

As the difference is zero, the original is divisible by 7 and by 13—and also by 11, though there is an easier test for this.

C. DUDLEY LANGFORD.

1624. *Integral cyclic figures.*

It will probably be as surprising to others as it was to me to find that cyclic figures with integral sides and diagonals are quite easy to make. A connection between Pythagoras' theorem and Ptolemy's theorem is less unexpected. I believe that the method is perfectly general, though it is most easily demonstrated by particular examples.

Take any two sets of Pythagorean numbers :

$$5^2 = 3^2 + 4^2,$$
$$13^2 = 12^2 + 5^2,$$

and combine them thus :

$$\begin{aligned} 65^2 &= 13^2 \times 5^2 \\ &= 13^3(4^2+3^2) = 52^2 + 39^2 \\ &= 5^2(12^2+5^2) = 60^2 + 25^2. \end{aligned}$$

For this idea, on which I have based the rest of this note, I am indebted to Sir Richard Harington.

Now combine the two right-angled triangles so obtained into one geometric figure in the four possible ways (Fig. 1) and calculate the

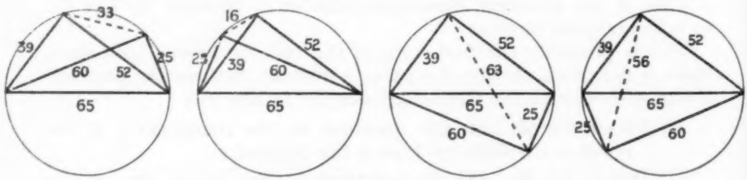


FIG. 1.

length shown in the figures by a broken line, by Ptolemy's theorem. This length must be rational. Now take these figures, 33, 16, 63, 56 and see which of them enable us to get new Pythagorean triangles.

$$65^2 = 33^2 + 56^2,$$
$$65^2 = 16^2 + 63^2.$$

It has previously been shown how the first of these may be used to make an all-integral cyclic hexagon (Note 1557).

From $$65^2 = 16^2 + 63^2 = 39^2 + 52^2$$

we can get a similar set of four figures, but only two give integral values for the new side, and these were known before,

$$65^2 = 25^2 + 60^2.$$

We can now get a new cyclic hexagon (Fig. 2).

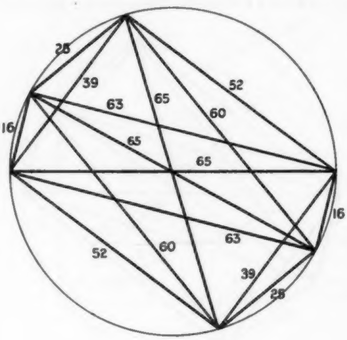


Fig. 2.

Next taking $$65^2 = 16^2 + 63^2$$
$$= 33^2 + 56^2$$

we get nothing of value.

Finally by taking

$$65^2 = 16^2 + 63^2$$
$$= 60^2 + 25^2$$

we again get $65^2 = 39^2 + 52^2$, and the cyclic hexagon just given.

Instead of combining two sets of Pythagorean numbers, we can use just one set in this way:

$$5^2 = 3^2 + 4^2 ;$$

$$25^2 = 5^2(3^2 + 4^4) = 15^2 + 20^2.$$

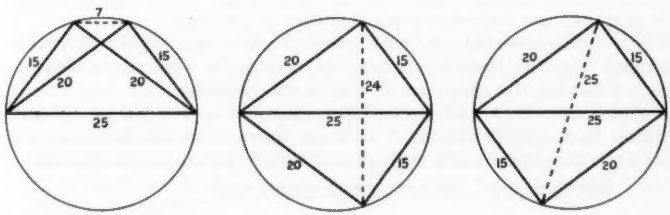


Fig. 3.

Again, we find numbers for a new Pythagorean triangle, $25^2 = 24^2 + 7^2$. These give the cyclic hexagon given in Note 1514.

Apparently the next smallest hypotenuse that can be made up in this way is 125.

$$\begin{aligned} 125^2 &= 5^2(25^2) \\ &= 5^2(15^2+20^2) = 75^2+100^2 \\ &= 5^2(24^2+7^2) = 120^2+35^2. \end{aligned}$$

By the methods already shown we get but one new expression for 125^2, namely $125^2 = 44^2 + 117^2$, and the cyclic hexagon of Fig. 4.

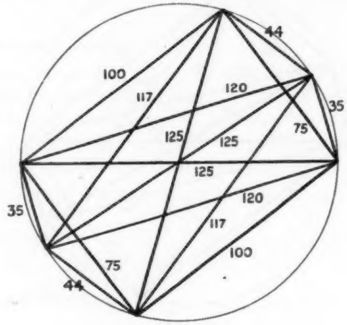


Fig. 4.

As we should expect, 169^2 works out in exactly the same way as 25^2:

$$\begin{aligned} 169^2 &= 13^2(13^2) \\ &= 13^2(12^2+5^2) = 156^2+65^2. \end{aligned}$$

This, by geometric treatment, using Ptolemy's theorem, gives

$$169^2 = 119^2 + 120^2$$

and this cyclic hexagon (Fig. 5) has the same type of symmetry as that obtained from 25^2 (Note 1514). The figures look rather different because the longer sides of the one correspond to the shorter sides of the other.

It seems that much of this work might be of interest to a good class, after Ptolemy's theorem had been done. The way in which Pythagoras and Ptolemy work together is amusing.

If a cyclic quadrilateral with integral sides, diagonals and area is wanted, say for Hero's formula, we can easily pick out a quadrilateral having the diameter as one of its diagonals.

These cases have all been fairly simple, but we can go on and obtain more complications if we need them. For example, we can take the lowest common multiples of two hypotenuses, in this way:

$$\begin{aligned} 325^2 = 5^2(65^2) &= 5^2(56^2+33^2) = 280^2+165^2 \\ &= 5^2(60^2+25^2) = 300^2+125^2 \end{aligned}$$

$$
\begin{aligned}
&= 5^2(52^2+39^2) = 260^2+195^2\\
&= 5^2(16^2+63^2) = 80^2+315^2\\
= 13^2(25^2) &= 13^2(20^2+15^2) = 260^2+195^2 \quad \text{(obtained before)}\\
&= 13^2(24^2+7^2) \;\; = 312^2+91^2.
\end{aligned}
$$

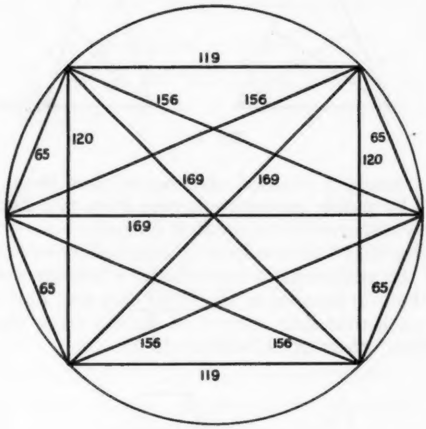


FIG. 5.

These, by treatment, yield new values for 325^2; some work can be saved by making use of the fact that the cyclic hexagons already given for 25^2 and 65^2 all have disguised 3, 4, 5 triangles.

$$
\begin{aligned}
325^2 &= 36^2 + 323^2\\
&= 115^2 + 220^2\\
&= 80^2 + 315^2.
\end{aligned}
$$

Is it true that "any Pythagorean angle can be expressed as the sum or difference of two other Pythagorean angles"? I was led to suspect it while doing this work.

C. DUDLEY LANGFORD.

1625. *Further notes on Apollonian figures.* (See Notes 1289, 1354.)
While working with Mr. Hamilton Dick's generalised form of my formulae for Apollonian numbers, I came across a few new points of interest. Some of these would be of particular use to Meccano enthusiasts since Meccano is, after all, only a matter of integral-sided girders. Puzzle makers and those wanting integral figures for class work or for examination purposes may very well find uses for these new figures.

The surprising thing which emerged from my investigation was

the large proportion of interesting cases that had come out from my original formula. But two extremely interesting figures were new (Fig. 1).

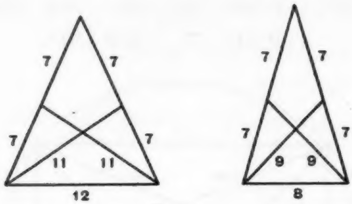


FIG. 1.

Curiously enough, I realised afterwards that these could have been deduced by purely geometric means from two figures which I had previously obtained from my first formula.

Remembering that "the medians of a triangle trisect one another" and that "if the midpoints of two sides of a triangle are joined, the line joining them is parallel to the third side and half its length", we can get an astounding variety of figures from the two given figures and from others given in Note 1289.

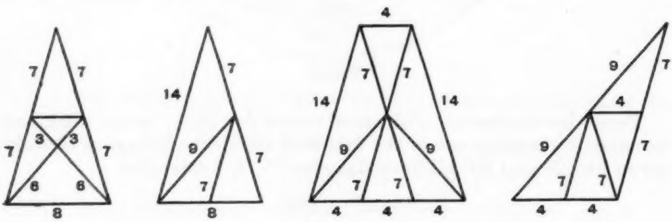


FIG. 2.

Meccano workers will easily continue these to give a useful construction for a long girder. C. DUDLEY LANGFORD.

1626. *Brianchon's Theorem.* (Cf. Note 1547, XXV, p. 247.)

Let $\Sigma=0$ be the tangential equation of a conic and $A=0$, $B'=0$, $C=0$, $A'=0$, $B=0$, $C'=0$ be the equations of the vertices of a hexagon circumscribed about the conic. If $P=0$ is the equation of a point, $P\Sigma=ABC$ is the equation of a curve of class there satisfied by the tangents to $\Sigma=0$ passing through A, B, C. P may be chosen so that $P\Sigma-ABC\equiv A'B'C'$. Thus $ABC=0$, $A'B'C'=0$ are satisfied by tangents to either $\Sigma=0$ or $P=0$. The sides of the hexagon account for six of the tangents to $\Sigma=0$ and so the lines $A=0, A'=0$; $B=0, B'=0$; $C=0, C'=0$ are concurrent in P. RAYMOND SMART.

CORRESPONDENCE.

STARRED QUESTIONS.

To the Editor of the *Mathematical Gazette.*

Sir,—To follow up Mr. C. V. Durell's suggestion in the *Gazette*, May 1942, p. 96, here are the two best questions that I can remember from scholarship papers of recent years.

(i) If the a's, b's, c's are positive and such that

$$a_1 > b_1 + c_1, \quad b_2 > c_2 + a_2, \quad c_3 > a_3 + b_3,$$

prove that the determinant $\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$ is not zero.

(ii) If a, b, c, d are positive integers and $bc - ad = 1$, prove that there is no fraction x/y between a/b and c/d such that y is less than $b + d$.

The quadrangle question quoted by Mr. Durell was very good ; perhaps it would have been better still if it had been the only question in a three-hour paper with an invitation to the candidates to supply two or more solutions.

Yours, etc.,

A. Robson.

A TRIPOS QUESTION.

To the Editor of the *Mathematical Gazette.*

Sir,—Those who intend taking the Mathematical Tripos may be comforted in a small measure by the following problem, which is a complete question from the paper of May 1894, and is easily the simplest I have come across in such examinations. The examinee is limited to the methods of pure geometry.

" A straight line drawn through the vertex of a triangle ABC meets the lines DE, DF, which join the middle point D of the base to the midpoints E, F of the sides, in X, Y ; show that BY is parallel to CX."

Proof. (I omit a figure, which readers can readily supply). Produce ED to meet BY in T. Since D, E, F bisect the respective sides of the triangle, ET is parallel to AB. Thus since $AF = FB$, then $XD = DT$. Hence in the triangles CXD, DTB, we have $CD = DB$ (given), $XD = DT$ (proved) and the vertically opposite angles XDC, BDT are equal. Thus the two triangles are equal in all respects and $\angle XCD = \angle DBT$; that is, CX is parallel to BT.

This must surely hold the record for simplicity, being hardly up to matriculation standard. Can any of your readers suggest what the examiners expected the candidates to produce?

Yours, etc.,

Ronald F. Newling.

SPELLING.

To the Editor of the *Mathematical Gazette.*

Sir,—I have just finished marking the scripts of about 300 candidates aged $13\frac{1}{2}$ years—*i.e.* near the normal age for leaving school. The subject was Geometry, but I was so impressed with the amount of bad spelling of mathematical terms that I venture to draw the attention of teachers to the state of affairs. Some of the words are admittedly awkward, but the boys had presumably been using them for months at least. Efforts like " *strait* " or " *circumpherence* " pale before " *pizaygarus therom* " ; but to emphasise my

point I give you the complete list for the word "*isosceles*". Figures in brackets denote frequency.

isoceles (12); *isocelles* (4); *isosoles* (4); *iscoseles* (3); *iscosceles* (3); *issocles* (2); *issosceles* (2); *isoscoles* (2); *isoscles*; *isoscales*; *isceles*; *isosiles*; *icosales*; *issoceles*; *isosoceles*; *isoscelles*; *isosclese*; *iccoseles*; *isocoles*; *iscosoles*; *iscolece*; *isocalees*; *icosseles*; *iscoceles*; *iscoscoles*; *iscoscilles*; *iscossiles*; *isoseceles*; *isosolies*; *isossceles*; *isosoleuses*.

So unfamiliar had it become that *isosceles* got into my original list.

Yours, etc.,

E. A. MAXWELL.

ENTRANCE SCHOLARSHIP EXAMINATIONS.

THE Mathematical Association has been asked, both in 1941 and in 1942, by the mathematical examiners in one of the groups of colleges to comment on the suitability of the papers set for entrance scholarships.

As meetings of our committees are not being held, it has been impossible to discuss the details as fully as we would have wished to do under normal conditions. Nevertheless certain members of the Sixth Form sub-committee of the Teaching Committee have sent their comments to the Chairman of the committee, who has thus been able to respond to the invitation, explaining at the same time to the examiners the limitations under which the replies were drawn up. And our suggestions have been welcomed.

It is probable that other members of the Association would like to discuss in the Branches, in the *Gazette*, or otherwise, the matters that have been raised in the sub-committee, such as:

(i) Is the present usual arrangement of four papers on Algebra-Trigonometry-Calculus, Geometry, Statics-Dynamics, and General Paper satisfactory? Is there a proper balance between the various subjects? Is the General Paper, half pure and half applied, satisfactory?

(ii) Are the questions of a sort not only to test ability but also to encourage the teaching of the right topics in schools?

(iii) In Geometry, how should the questions be distributed over the various parts of the subject? Should indications be given more frequently about the kind of method expected? Is there enough, or too much, homography?

(iv) Are any changes of a general nature called for in the Mechanics? Should candidates be given the chance occasionally to show knowledge of physical subjects, *e.g.* in the General Paper?

(v) How should calculus be examined? Is it agreed that the fundamental elementary ideas about limits and convergence ought to be taught in schools?

(vi) Is there any suggestion that can be made for improving the liaison between school and universities in the matter of the scholarship examinations?

A. R.

1411. SUPER-ARITHMETIC.—*Wife*: "The price of the alarm clock was \$1.50, but I got a discount, so it only cost me 98c."

Husband: "Yes, but you know very well you could have got the same thing at Brown's for 75c."

Wife: "That may be, but then Brown's wouldn't have taken anything off."—*Good Hardware.* Quoted in *Literary Digest.* (Per Mr. T. R. Dawson.)

REVIEWS.

The Reliability of Mental Tests. By G. A. Ferguson. Pp. x, 150. 6s. 1941. (University of London Press)

Dr. Ferguson is a young scholar from Nova Scotia who has been working with Moray House, Edinburgh, since 1937. He has been able to have the guidance of Godfrey Thomson and to use some of the vast amount of material obtained by the routine of administering something like one-sixth of a million Moray House tests per annum, to deal, in this book, with some problems alike of theory and of practice of mental testing. It is a valuable and interesting book. There are, unfortunately, a number of misprints—as a rule not likely to mislead the careful reader—a few vaguenesses, incompatibilities and inconsistencies in the tables and the text, and a fair number of slips in the references to other literature, but these are no doubt fully explained by the fact that owing to the War Dr. Ferguson has not been able to revise his manuscript, which is dated 1940, or to see his book through the press.

A test, like an examination, is a measuring instrument : it differs from the latter in that it is composed of unit items, dichotomously scored into the alternative categories 1 and 0 respectively. Like all measuring instruments it is not perfect. Psychologists recognise two main sources of imperfection, though each of them occurs in cases even of physical measurements made indirectly.

(1) Invalidity—the test fails in sampling the candidates' traits as evidenced by abilities in particular directions. This, as has been pointed out elsewhere (*Brit. Journal of Educ. Psychology*, 1935, Ju., V (ii), pp. 180–193), includes the two features due to (*a*) bias and (*b*) smallness, in addition to the difficulty of arguing from traits to abilities (for example, it is commonly but erroneously assumed that if the latter is normally distributed the former is so also) and from abilities to traits (we can be able to do a thing by exercising various differing qualities, and by different approaches) and to Hamilton's "Faulty Indication".

(2) Unreliability—the test fails to measure that sample of abilities which it purports to measure. Dr. Ferguson, following Thouless and others, analyses this into (*c*) "function fluctuation" ("quotidian variability", "instability", etc., as well as variations due to "set" and "incentive" and other factors of "temporal contiguity"), this being a feature of the "unreliability of persons", and (*d*) reliability of tests (the accuracy with which a test measures the abilities which it measures when it measures them). When the item is included in a test and is marked right or wrong, this (*d*) surely includes at least two distinct notions :

(*d*, i) each item is given its proper weight by a sound formula ;

(*d*, ii) the marking schedule is accurately and consistently applied.

(*e*) There is a further aspect of unreliability at which Dr. Ferguson hints—the vague relationship that exists between ability, guessing and correctness of response.

Several methods have been proposed for measuring reliability. All those that have been used in practice are based on correlation. It has been the practice, followed by Dr. Ferguson, to correct for "attenuation". But this, as it is an attempt to adjust for the length of the test, is surely, if justified at all, a correction for validity, and not, as psychologists claim, for reliability. He also uses a correction for selection which seems to be appropriate only under special conditions. Reliability has often been measured by the correlation between two different series of observations of what are assumed to be the same mental abilities. Dr. Ferguson finds reliability coefficients for Moray

House Intelligence Tests of ·9372 by correlating the I.Qs. of 1030 eleven-year old Doncaster children as estimated by two tests taken at two testings seven weeks apart, and values somewhat higher from another experiment (the West Riding Enquiry of 1937 : three I.Ts., 39 schools, 1535 children : why did not the enquiry rotate the tests among the schools in the six possible ways?). The correlations here given involve at least the three features *c*, *d* i, *d* ii above, and thus do not appear to be a final estimate of the reliability of the test as apart from the reliability of persons. It may be noted that in spite of the high reliability of tests and consistency of candidates found, the S.D. of the Test-Retest difference is some 4, 5, or even 6 points I.Q.

Dr. Ferguson gives an interesting survey of the method depending on answer pattern which is being developed at Moray House. It is a very promising method, but it is not clear that reliability might not have a definite meaning apart from discriminating power : it is not clear in this connection what is meant by the reliability coefficient of an item of a test. In Chapter VI Dr. Ferguson develops certain corollaries of the answer pattern method. An answer pattern is unique if the score x of any person tested is composed of correct responses to the x easiest items on the test ; the greater the lack of uniqueness in the answer pattern matrix the lower the reliability of the test. (It may be noted, in passing, that the simple rules of algebra would not suffice to enable our pupils to understand, for example, the references to matrices on pages 21 and 52, in spite of the claim in the preface.) Some of the conclusions are :

(*a*) we should expect the reliability of tests to increase with increase of age ;

(*b*) with a unique answer pattern matrix if the items are maximally different from one another with respect to difficulty we have a rectangular distribution of raw scores ;

(*c*) high test-discrimination-power involves high inter-item-correlation ;

(*d*) a test yielding a platykurtic (flat-peaked) distribution of raw scores is the more discriminating ;

(*e*) a U-curve with an anti-mode dividing the range in the ratio $a : b$ will best discriminate at a level of $b : (a+b)$ of the way down the range.

In the exposition of the experimental work, Dr. Ferguson refers to certain aspects already mentioned. He found that the Moray House Tests that he used seemed to be more reliable, and the I.Q. less variable, at the lower than at the higher ranges of ability. He found that these group tests were at least as reliable as the individual tests (such as the Stanford Binet, new Terman Revision). The Doncaster experiment worked also with some Moray House Arithmetic Tests. These are of 102 items (42 mechanical, 60 simple problems) administered in thirty minutes. The S.D. in A.Q. differences is of order 4. Reliability is of order ·96, and is greater than for I.Ts. or for English Moray House Tests. The greater reliability is supposed to be due to the greater discriminating power of the A.T. items. As a corollary he concludes that a good test will have items that correlate as highly as possible with the criterion and at the same time as highly as possible with each other. A stimulating book.

FRANK SANDON.

Applied General Statistics. By F. E. CROXTON and D. J. COWDEN. Pp. xviii, 944, xiii. 20s. (Pitman)

This book is an exhaustive volume—shall we say an enlarged and American Bowley?—by the Associate Professor of Statistics at Columbia University, and the Associate Professor of Economics, School of Commerce, at the University of North Carolina. It is undated, but references include (p. 80) a provisional death rate for 1938, whilst the last edition of Fisher mentioned is the seventh (in one place, p. 325, dated as 1936, but elsewhere as 1938). The

writers have already (1934, New York) published ***Practical Business Statistics***, and the present volume, although covering very fully most of the traditional course, exhibits distinctly the authors' main interests. Of 25 chapters (referred to indiscriminately by Roman or Arabic notation) time series, index numbers, forecasting and allied topics take up almost one-third. But even this leaves us with over 600 pages of text and 18 appendices, these latter taking up alone more than 100 pages and including a number of useful tables. It was, perhaps, hardly necessary to include a table of sines and cosines of 0° (1°) 90°. Appendix O is apparently printed from a worn plate. But on the whole the book is extremely well and carefully printed. The reproduction of the U.S.A. Census of Population Schedule on p. 20 is in excessively small type, conflicting with the rule laid down explicitly on p. 65. There are a very few misprints. The volume should prove very useful as a reliable text and reference book, primarily for teachers in economic statistics, though it should answer many statistical questions for other workers. The educationalist will note the omission of factor analysis, reliability and attenuation, and the agriculturalist of factorial analysis and of Latin Square, whilst even the economist will find that modern methods of market research are not dealt with. The indications given casually (pp. 634, 642, 644) of free statistical services in the U.S.A. are rather surprising to a foreigner. The authors are obviously very experienced teachers and their expositions are good and clear and their approach sound and useful. It is, however, a bit awkward to find in the chapters on trend and index-numbers that correlation, though needed, is not dealt with till later. There are very full references to the sources of tabulated material and of formulae, though it is hardly going to the fount of information to quote (p. 511) as authority for dates of Easter a publication of the *New York World Telegram*. The treatment is said to be at an elementary level : our comment on this must be that everything that is at an elementary level is here, so that the book at twenty shillings is very good value.

The references are practically all American, though frequent reference is made to the standard books of R. A. Fisher, Yule and Kendall, and Tippett, and to the work of Karl and Egon Pearson. There is even something about Gram-Charlier, Tchebycheff and Camp-Meidell. It is strange to find Stamp given (p. 161) as authority for the story of Randolph Churchill's damned dots. There are full proofs of a number of formulae and full expositions of a number of calculations, often by alternative methods, so that they throw much light for the student on the meaning of the analysis. The mathematics is not of the highest standard of rigour : thus we have the binomial theorem stated (p. 922) in a form appropriate when m is a positive integer, without the limitation being stated. Numerical constants are given to a certain number of figures with no indication that the value is not exact. The setting-out of the proof on p. 223 would be heavily penalised by any member of the M.A. "Obviously"—as usual—glosses (p. 208) a step of dubious validity. The statement about rounding (p. 150) is incorrect. No distinction is drawn between rates and ratios. I do not think that 4·5! is an accepted notation for $4{\cdot}5 \times 3{\cdot}5 \times 2{\cdot}5 \times 1{\cdot}5 \times 0{\cdot}5 \times \sqrt{\pi}$ (p. 326). There are vague, loose or inaccurate phrases in a number of places (for example, pp. 173, 188, 218, 221, 222, 230, 293, 310, 363, 440, 461, 498, 580, 590, 742, 858). One turns with interest to the full and useful glossary at the end (Appendix Q). The entry of "infinity" is a neat solution of a difficulty; we find simply the entry ∞ (p. 926).

The section on graphs is very full and careful. It is unfortunate that the student is explicitly told (p. 85) that the plotted points need not be shown (even on p. 442 where they are said to be shown by large dots they seem to have been lost in the reproduction of the figure), whilst he is not told whether to draw the curve smooth among the points or spiky through the points. The

chief criticism is, however, that although there are over 250 graphs, numbered serially, yet there is no index : the page references are not usually given, so that it is a tedious task to read sections where the charts are much used, for example in chapters 4, 5 and 6.

The section on sampling is not altogether satisfactory : no distinction is drawn between unbiassed and random sampling, whilst the method of random sampling (p. 307) given by the authors suggests that random sampling numbers are unknown. Representative sampling is undefined (p. 584), but apparently it is stratified (p. 586). There is an appendix (C) on calculating machines : the slide rule appears here, but the authors are optimistic in thinking that 3- or 4-figure accuracy can be obtained with a 10-inch rule (p. 870 ; the illustration on the very next page gives this the lie), whilst they choose to use logarithmic paper (p. 121) instead of a slide rule to get easily a logarithmic scale of any stretch. In addition to the calculating machines some other apparatus is referred to : the mechanical device on p. 268 for obtaining a binomial distribution is very neat. So is the use of a bread knife for drawing a serrated line (p. 82). There is no reference to the cope-chat (paramount) cards—the Findex (p. 44 ; not, I believe, on the English market) does not seem such a good idea. There is no indication that there is a mechanical sorter-tabulator as well as an electrical one (p. 41).

It will be seen, therefore, from the criticisms above, that the book has ranged over an immense field, but that the faults, from the point of view of the average student, are only of a relatively minor character. The book can whole-heartedly be recommended as an encyclopaedic and reliable outline of modern statistical practice and of most aspects of its elementary theory.

FRANK SANDON.

A Treatise on Algebra. I. Arithmetical Algebra. Pp. xvi, 399. **II. On Symbolical Algebra and its Applications to the Geometry of Position.** Pp. x, 455. By G. PEACOCK. $6.50. 1940 : reprinted from the edition of 1842–5. (*Scripta Mathematica*, Yeshiva College, New York)

George Peacock (1791–1858), a member with Babbage and Woodhouse of the Analytical Society which did so much to rouse British mathematics from eighteenth-century slumber, may be reckoned, if not one of the greatest of British mathematicians, at least one of the most influential. Not only did he play a part in making this country aware of the great mathematical strides taken in France in the late eighteenth century ; he was one of the first mathematicians to perceive that pure algebra should be a deductive system based on axiomatic foundations, and his steps were among the first to tread—however vaguely and uncertainly—on the path which has led to the present-day domain of algebra, where the severely logical treatises of van der Waerden and Albert are conspicuous landmarks.

Much of Peacock's algebraic work must now seem valueless or misleading. For instance, many hard things have recently been said about his *Principle of the permanence of equivalent forms*. A glance at his chapter on the binomial theorem for a general index will show at once how worthless some of his applications of this principle could be. Many clear-headed students must have been worried by " Principles ", this of Peacock's, that of " Duality " and that of " Continuity ", and been inclined to suppose that such " Principles " were either means of asserting the truth of statements which to common sense were obviously false, or of " getting away " with some result which ought to be true but could not be proved. Mathematics is, or ought to be, unprincipled. But Peacock's principle did at least serve him as a guide towards a sound treatment of, for example, a^n when n is not a positive integer.

If we recall that Peacock was not only Lowndean Professor of Geometry at Cambridge (1836–1858), but also (1839–58) Dean of Ely (and a collaborator with George Gilbert Scott in the work of restoring the Cathedral fabric), we may be less surprised at his acceptance and construction of dogma and content ourselves with wondering at his versatility.

In the first volume, Peacock deals with what he calls " arithmetical algebra " as distinct from " symbolic algebra ". Thus the symbolism must be immediately concrete, and $a-b$ is an *impossible* operation if b is greater than a. The scope of the first volume is nevertheless wide, embracing scales of notation and simple propositions in the Theory of Numbers. Much of it, however, is elementary. We can measure the improvement in teaching by noting such horrid examples as " By selling a horse for £24, I lose as much per cent. as the horse cost me. What was the prime cost of the horse? " ; and we can catch echoes of a more spacious age on p. 269 : " Two porters A and B drink from a cask of beer for 2 hours, after which A falls asleep and B drinks the remainder in 2 hours and 48 minutes ; but if B had fallen asleep and A had continued to drink, it would have taken him 4 hours and 40 minutes to finish the cask. In what time would they be able to drink it separately? "

The second volume extends the operations of arithmetical algebra to general symbols. A good deal of analytical trigonometry is included, with the introduction of $\sqrt{-1}$; and the theory of equations is considered. It is interesting to note that the idea of the sine and cosine as ratios was still novel at the time of publication of this treatise.

This reprint will not interest those who regard teaching as simply the methodical presentation of a static body of knowledge. To the true teacher, who knows that his subject is alive, changing and growing even as he tries to describe some small part of it, Peacock's work will provide much food for thought. It is a landmark in the teaching of algebra, and should apply a stimulus to those who read it to-day and note how far behind us it sometimes appears to be and at other times how curiously close.

We may hope that other classic treatises may be revivified by such reprints. For this one, we congratulate the collaborating bodies,. St. John's College, Annapolis, and *Scripta Mathematica*, on their production. To conclude, we may re-quote from an un-named writer (surely De Morgan) in the *Athenaeum* (1858, ii, p. 650) already quoted in the *Gazette* (XV, p. 159, Gleaning 775) : " We cannot undertake to describe in full what [Peacock] did for Algebra. . . . If his opinions do not find active and successful supporters, in twenty years Oxford will be the great school of the exact *disciplines* in England, and Cambridge will be but the Epsom or the Doncaster of bookwork and problem races." T. A. A. B.

Portraits of Famous Physicists. With biographical accounts by HENRY CREW. $3.75. 1942. (*Scripta Mathematica*, Yeshiva College, New York)

Notices in the *Gazette* (XX, 352 ; XXII, 320) have already commented on the two excellent *Scripta Mathematica* portfolios of portraits of mathematicians. Now the physicists have their turn ; the selected twelve are Galileo, Huygens, Newton, Ampère, Fresnel, Faraday, Joule, Clausius, Maxwell, Gibbs, Hertz and Rowland. The selection was made in 1937, and the committee of selection decided to omit portraits of living physicists, a decision which explains the absence, for example, of Rutherford. Biographical notes, limited to 800 words in each instance, have been given by Henry Crew.

The reproductions are excellent ; that of the Phillips portrait of Faraday, in the National Portrait Gallery, is particularly charming. T. A. A. B.

A Royal Road to Geometry. By L. Leland Locke. Reprinted from *Scripta Mathematica*, Vol. VIII, No. 1, pp. 34–42. March 1941. 15 cents. (New York)

Thomas Malton (the elder) taught Geometry in London in the second half of the eighteenth century, promising to do this " in less than half the time usually spent in it ". He deliberately sets out by a combination of practice and theoretical procedure to get away from the rigorous Euclidean methods ; the full significance of such efforts, however, can become clear only when standard textbooks of the time are compared with this " Royal Road ". In solid geometry Malton's diagrams can be bent up into three-dimensional space (following Billingsley's edition of Euclid of 1570). Malton is better known as the author of a treatise on Perspective, and some remarks in connection with this seem rather interesting to us to-day : " of all the calamitous accidents which of late years have happened in this Metropolis none has attracted the attention of the public more than the Fire, at Messrs Cox & Bigg's, Printers, in the Savoy . . . on account of the many valuable literary Productions which were destroyed there . . . amongst the rest was the Compleat Treatise on Perspective in folio . . ." and later, complaining about plagiators, Malton says : " a quarto work has been published, since the commencement of the year 1793, by a Mr Sheraton, Cabinet Maker. . . ." A. P.

Mary Somerville. By A. W. Richeson. Reprinted from *Scripta Mathematica*, Vol. VIII, No. 1, pp. 5–13. March 1941. 15 cents. (New York)

The author has neither read Mary Somerville's autobiography nor even looked up the article on her life in the *Dict. Nat. Biogr.* A full-page portrait is given which is a poor drawing made, it would appear, after a bad reproduction of the crayon by J. Swinton, though no artist's name is given. The article is full of errors in actual facts, and of misstatements in general remarks at almost every point. The birthdate is wrong, " Naysmith " is the famous artist Nasmyth ; " Dr Broughton " means Lord Brougham ; also, the author pretends to have made " a careful search ", unsuccessfully, of the *Quarterly Review* 1827–1837 to find an article by Mary Somerville, and pronounces a conjecture to explain why other " biographers have been misled " : but the article did appear in vol. 55, 1835, Dec., pp. 195–233. . . . There is, in fact, hardly a line in the whole article that will bear scrutiny. It would, of course, not be worth while to mention this if the article had not appeared in a periodical of reputation, and one must wonder at the state of mind of the author who dared to offer such a piece of work for publication. History of Mathematics by biography has of late become quite a fashion, but most of us believe that this does not imply any licence with regard to accuracy and reliability. A. P.

Charles Sanders Peirce as a Pioneer. By C. J. Keyser. Pp. 24. 25 cents. 1941. (*Scripta Mathematica*, Yeshiva College, New York)

In this lecture delivered in 1935 Keyser gives a biography of the " Father of Pragmatism " and, in a manner intelligible to the non-specialists, he illustrates Peirce's contributions to modern logistics ; some lines of thought with which we are familiar through Russel's *Principles* are shown to originate from Peirce's researches. An edition of Peirce's collected works is appearing at the Harvard Press. Six volumes have already been published. A. P.

Waring's Problem : Squares. By Ralph G. Archibald. Reprinted from *Scripta Mathematica*, Vol. VII, Nos. 1–4, pp. 33–48. 1940. 15 cents.

Interest in the history of Mathematics is frequently first roused when one attempts to collect and compare the endeavours of prominent scientists of the past to solve one particular problem ; Archibald has chosen a particular

form of Waring's Problem and compiled the contributions, with and without proof, of mathematicians from Fermat's time on with regard to this problem; this culminates in a proof of the theorem that every integer is the sum of not more than four squares. The author is particularly conscientious in giving all quotations both in the original and in an English translation; this pleonasm is, however, led *ad absurdum*—when Landau's proof is finally given in full (literal) translation without the German text in the "footnotes": history, it seems, ends with Euler! A. P.

Natural Numbers as Ordinals. By ABRAHAM ADOLF FRAENKEL. Reprinted from *Scripta Mathematica*, Vol. VII, Nos. 1–4, pp. 9–20. 1940. 15 cents.

This is an introduction into the problems of number-axiomatics, written in a lucid manner "for the common reader". Peano's axioms are set out and their independence demonstrated; there is a chapter on mathematical induction and the "happy circumstance" of the correspondence of ordinals and cardinals for finite sets is analysed as being capable of proof. A. P.

Counterpoints and associated cubic curves. By J. M. FELD. Pp. 6. 10 cents. Reprinted from *Scripta Mathematica*, VII, pp. 113-118. 1940. (*Scripta Mathematica*, Yeshiva College, New York)

Counterpoints are isogonal conjugates viewed projectively. A simple definition relative to a basic triangle and to two basic points A, B leads to the connecting equations

$$\rho x_1 y_1 = a_1 b_1, \quad \rho x_2 y_2 = a_2 b_2, \quad \rho x_3 y_3 = a_3 b_3.$$

Some cubic curves which are not self-corresponding under the counterpoint transformation are studied, and there are some remarks on Schröter's ruler construction for cubic curves. D. P.

Post-Certificate Mathematical Tests. By L. MALONE. Pp. iv, 64, 21, 2. 2s. 1942. (Macmillan)

It is a pleasure to quote from this slim volume two questions:

Test 33, No. 2.

"If it is possible, sum the series

$$\tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}) + \dots, \quad \text{and} \quad \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + \dots."$$

Ans. Impossible.

Test 21, No. 7. "Correct any error:

$$\int (1+2x)^2 dx = \int (1 + 4x + 4x^2) dx = x + 2x^2 + \frac{4x^3}{3} + c;$$

alternatively,

$$\int (1+2x)^2 dx = \int (1+2x)^2 d\frac{(1+2x)}{2} = \frac{1}{2} \cdot \frac{(1+2x)^3}{3} + k."$$

Ans. None; but $c = \frac{1}{6} + k$.

The booklet certainly carries out the intention of its author to prepare pupils for any H.S.C. Examination and "to at last envisage Scholarship work": it is so easy to make recommendations as to how this should be done and so difficult to do it. There are 60 test papers, 30 for the first year and 30 for the second. There are 21 pages of answers and 2 for notes. At the beginning there are suggestions for schemes of work for the first and second years.

Here we have a good job well done. N. M. G.

Sure Foundation Arithmetic. By J. G. LOCKHART and T. M. YOUNG. Book 3 : pp. 104. 1s. 3d. 1941. Answers : pp. 48. 1s. Book 4 : pp. 159. 1s. 4d. 1940. Answers : pp. 64. 1s. Book 5 : pp. 176. 1s. 6d. 1940. Answers : pp. 64. 1s. Book 5A : pp. 96. 2nd impression. 1s. 1941. Answers : pp. 25. 9d. (Craig & Wilson, Glasgow)

The books cover the work of a Junior School, with the last book giving further practice in general principles and with especial stress on problems.

The most immediately striking features are the attractive lay-out, excellent type and clear setting-out. The print is a sensible size to obviate eye-strain, and all the exercises are most clearly arranged. I do not think that I have seen any other arithmetic book, or indeed mathematics book, with such a high standard in these respects.

These books assume an average pupil. There are exercises labelled Mental, Mechanical and Problem. There are plenty of examples of a similar type to ensure sufficient practice in the principle or table involved.

The books strike one or two irritating notes ; for example, the use of the jarring word " Revisal " in place of the usual " Revision ". There is also a complicated table meant to demonstrate the mutual relations between the parts of £1. Probably the child would scarcely glance at it, as it appears to be quite beyond a young child's power to appreciate the significance of diagrams.

These arithmetic books represent a sound balance between the older type of " exercises " and the modern type of " interesting examples ". D. E. S.

BOOKS RECEIVED FOR REVIEW.

P. F. Burns. *First steps in astronomy without a telescope.* Pp. ix, 214. 5s. 1942. (Ginn)

R. Courant and **H. Robbins.** *What is Mathematics?* Pp. xix, 521. 25s. 1942. (Oxford)

H. S. M. Coxeter. *Non-euclidean geometry.* Pp. xv, 281. $3.25. 1942. Mathematical Expositions, 2. (University of Toronto Press)

W. G. Emmett. *An inquiry into the prediction of secondary-school success.* Pp. 58. 2s. 1942. (University of London Press)

H. L. Jones. *School certificate arithmetic. A revision course.* Pp. viii, 100. 2s. 6d. 1942. (Arnold)

H. Levy. *Elementary mathematics.* Pp. 216. 5s. 1942. (Nelson)

W. O. Manning. *How aeroplanes fly.* Pp. 64. 2s. 1942. (Oxford)

B. Segre. *The non-singular cubic surfaces.* Pp. xi, 180. 15s. 1942. (Oxford)

L. Turner. *General mathematics. II.* Pp. iv, 287. 5s. 1942. (Arnold)

D. V. Widder. *The Laplace transform.* Pp. x, 406. 36s. 1941. Princeton Mathematical Series, 6. (Princeton University Press ; Humphrey Milford)

M. Williams. *Air navigation plotting.* Pp. 88. 2s. 6d. 1942. (Bell)

1412. " The American 14-inch [naval gun] fires a 1400 pound shell, the 16-inch a 2100 pounder. Each ship thus fires a total of 16,800 pounds per broadside. But the 16-inch, a heavier shell, fired with a larger charge, arrives at greater speed and thus can penetrate a greater thickness of armour. Moreover, the thickness of the shell itself—the metal case in which the bursting charge is enclosed—is practically the same in both shells. The contents—the explosive charge—thus get all the increase in the size of the shell. Since the object under consideration is roughly a cylinder, the bursting charge thus increases relative to the square of the weight of the shell.

" Square the 1400 pounds of a 14-inch shell and the result is a reference figure of 1,960,000. Square the 2100 pounds of a 16-inch and the comparative reference figure is 4,410,000. Twelve 14-inch shells thus give a total punch of 23,520,000 in this system of reference figures ; eight 16-inch deliver 35,280,000. . . ."—Fletcher Pratt, *Saturday Evening Post*, October 7, 1939, pp. 9 and 90, article entitled *Columbia the Gem of the Ocean*. [Per Mr. D. D. Kosambi.]

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